Exploring Relational Structures Via $\mathcal{FLE}$

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Abstract. Designing ontologies and specifying axioms of the described domains is an expensive and error-prone task. Thus, we propose a method originating from Formal Concept Analysis which uses empirical data to systematically generate hypothetical axioms about the domain, which are represented to an ontology engineer for decision.
In this paper, we focus on axioms that can be expressed as entailment statements in the description logic $\mathcal{FLE}$. The proposed technique is an incremental one, therefore, in every new step we have to reuse the axiomatic information acquired so far. We present a sound and complete deduction calculus for $\mathcal{FLE}$ entailment statements.
We give a detailed description of this multistep algorithm including a technique called empirical attribute reduction and demonstrate the proposed technique using an example from mathematics.
We give a completeness result on the explored information and address the question of algorithm termination. Finally, we discuss possible applications of our method.

1 Introduction

When designing systems for knowledge representation and exchange (such as expert systems, semantic web applications, ontologies in general, etc.) one central task is to specify not only the basic terms used to characterize the entities of the described domain but also the logical interrelationships between them. This information (called domain axioms or rules) encodes the background or world knowledge and enables automatic reasoning about the domain.

Since the system's knowledge has to match reality, this specification task can not be carried out fully automatically, unless one has already a complete representation of the part of the world to be described. Otherwise human assistance is necessary. Nevertheless, also incomplete data about reality maybe extremely helpful in order to reduce the set of possible axioms in advance (by assuming their consistency with the data).

In this paper (extending our former publication [13]) we present an algorithm that helps to determine all domain axioms of a certain logical shape by successively presenting questions to an expert. This is done in a way, such that no

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redundant questions will be asked. Additional information (concerning entities or a priori known rules of the domain) are taken into account.

Section 2 will shortly recall some notions of formal concept analysis (formal context, implication, stem base) as far as they are needed for an understanding of the attribute exploration algorithm.

In Section 3, based on Description Logic (DL) a class of concept descriptions ($\mathcal{FLE}$) is defined together with an extensional semantic using binary power context families. Definitions of entailment and equivalence of that formulae with respect to a fixed semantics are discussed.

In Section 4, we define a special kind of formal contexts that can be constructed on the basis of a binary power context family from a set of DL-formulae. We observe that implications within such a formal context correspond to valid DL entailment statements.

The algorithm, that consists of a sequence of exploration steps is described in Section 5: initialization, the actual exploration step yielding a stem base $\mathfrak{B}_i$, and how the stem base can be used to determine the attribute set and background knowledge for the next exploration step.

Section 6 discusses how the validity of an arbitrary entailment statement between concept descriptions from $\mathcal{FLE}_i$ can be decided using just the stem bases $\mathfrak{B}_0, \ldots, \mathfrak{B}_i$ obtained by the exploration process.

Section 7 addresses the question under which conditions the proposed algorithm terminates, i.e., in which cases a complete information acquisition is achieved. We will demonstrate how any $\mathcal{FLE}$ formula can then be decided based on the explored knowledge.

In Section 8, we apply the presented algorithm to an example from basic mathematics.

Concluding, in Section 9 we discuss how this algorithm can be applied e.g. for generating and refining ontologies.

2 Attribute Exploration

Here, we will introduce notions from Formal Concept Analysis relevant for this theory. For a comprehensive introduction into FCA cf. [6].

Definition 1. *A* FORMAL CONTEXT $\mathbb{K} := (G, M, I)$ *consists of two arbitrary sets G (the elements of which are called objects) and M (the elements of which are called attributes) and a relation $I \subseteq G \times M$. The incidence gIm for $g \in G$ and $m \in M$ is read as "object g has attribute m".*

Definition 2. *Let M be an arbitrary set. If A and B are two sets with $A, B \subseteq M$ we will call the pair (A, B) an* IMPLICATION *on M. To support intuition we will write it as $A \to B$ in the sequel. We say an implication* HOLDS *for an attribute set C, iff from $A \subseteq C$ follows $B \subseteq C$. Moreover, an implication* HOLDS *in a formal context $\mathbb{K} = (G, M, I)$ iff it holds for all its object intents $g^I := \{m \in M \mid gIm\}$.*

Given a set $A \subseteq M$ and a set $\mathfrak{I}$ of implications on M, we write $A^{\mathfrak{I}}$ for the smallest subset of M which

– *contains A and*
– *fulfills all implications from $\mathfrak{I}$.*[1]

Let imp($\mathbb{K}$) *denote the set of all implications holding in* $\mathbb{K}$. *A set of implications* $\mathfrak{B}$ *is called* IMPLICATION BASE *of* $\mathbb{K}$ *iff it is*

– *complete, i.e.,* $A^{\mathfrak{B}} = A^{imp(\mathbb{K})}$ *for all* $A \subseteq M$ *and*
– *irredundant, i.e., for every implication* $i \in \mathfrak{B}$ *there is an* $A \subseteq M$ *with* $A^{\mathfrak{B}\setminus\{i\}} \neq A^{imp(\mathbb{K})}$.

Guigues and Duquenne [8] found a characterization of a canonical minimal implication base - the so called *stem base*. Ganter's attribute exploration algorithm [5] is an interactive method to determine the implication base of a formal context not entirely known in the beginning. The algorithm systematically presents potential implications (i.e., such ones that do not contradict the known part of the context) asking for their overall validity. A domain expert then has to decide: either (s)he confirms the implication - in that case it will be incorporated into the implication base - or he denies it - then he has to state a counterexample object which will be added to the considered context. This process continues, until the implications of the (still partial) context are just those mediated by the generated implication base. Figure 1 shows a scheme of the algorithm.

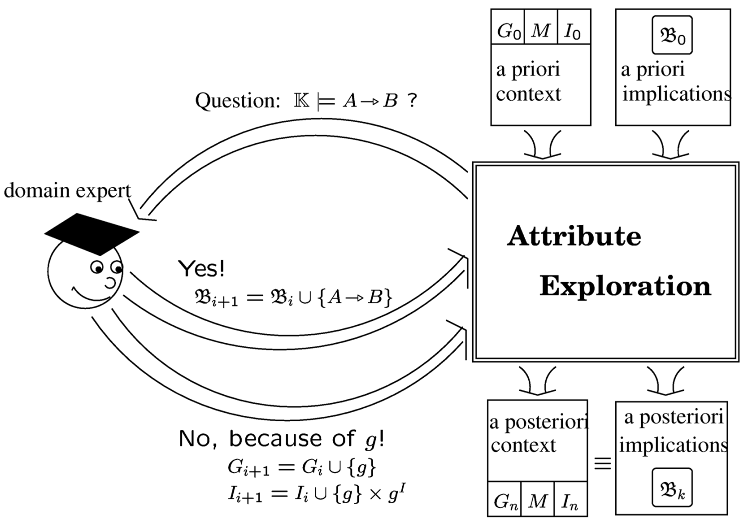

Fig. 1. Scheme of Ganter's attribute exploration algorithm.

[1] Since those two requirements are preserved under intersection, the existence of a smallest such set is assured. Moreover, note that the operation $(.)^{\mathfrak{I}}$ is a closure operator on M. Note also that, given A and $\mathfrak{I}$, the closure can be calculated in linear time (cf. [4]).

3 The Language $\mathcal{FLE}$: Syntax and Semantics

In this section we will introduce the Description Logic $\mathcal{FLE}$. We will just introduce the notions needed for our purposes, for a comprehensive overview see [2]. At first we will define the set $\mathcal{FLE}$ of all concept descriptions:

Definition 3. *Let $M_\mathcal{C}$, $M_\mathcal{R}$ be arbitrary finite sets, the elements of which we will call concept*[2] *names and role names, respectively. By $\mathcal{FLE}(M_\mathcal{C}, M_\mathcal{R})$ (or shortly: $\mathcal{FLE}$ if there is no danger of confusion) we denote the set of formulae (also called* CONCEPT DESCRIPTIONS*) inductively defined as follows:*

$$\begin{aligned} M_\mathcal{C} \cup \{\top, \bot\} &\subseteq \mathcal{FLE}, \\ \varphi, \psi \in \mathcal{FLE} &\Rightarrow \varphi \sqcap \psi \in \mathcal{FLE}, \\ \varphi \in \mathcal{FLE}, r \in M_\mathcal{R} &\Rightarrow \exists r.\varphi \in \mathcal{FLE}, \\ \varphi \in \mathcal{FLE}, r \in M_\mathcal{R} &\Rightarrow \forall r.\varphi \in \mathcal{FLE}. \end{aligned}$$

By $\mathcal{FLE}_n$ we denote the set of all concept descriptions from $\mathcal{FLE}$ with role depth of at most n.

Next, we will define what is an interpretation of $\mathcal{FLE}$. Sticking to the way of defining relational structures usual in FCA (see also [16]) we call it binary power context family. The common definitions in DL and modal logics (see e.g. [3]) are just syntactic variants thereof.

Definition 4. *A* BINARY POWER CONTEXT FAMILY *on a set Δ, called the* UNIVERSE*, with $\Delta \neq \emptyset$ is a pair $(\mathbb{K}_\mathcal{C}, \mathbb{K}_\mathcal{R})$ consisting of the formal contexts $\mathbb{K}_\mathcal{C} := (G_\mathcal{C}, M_\mathcal{C}, I_\mathcal{C})$ and $\mathbb{K}_\mathcal{R} := (G_\mathcal{R}, M_\mathcal{R}, I_\mathcal{R})$ with $G_\mathcal{C} = \Delta$ and $G_\mathcal{R} = \Delta \times \Delta$.*

As we know from the definition of formal context, $M_\mathcal{C}$ and $M_\mathcal{R}$ are arbitrary sets and $I_\mathcal{C} \subseteq G_\mathcal{C} \times M_\mathcal{C}$ as well as $I_\mathcal{R} \subseteq G_\mathcal{R} \times M_\mathcal{R}$.

Definition 5. *The semantical mapping $[\![.]\!]_{\vec{\mathbb{K}}} : \mathcal{FLE}(M_\mathcal{C}, M_\mathcal{R}) \to \mathcal{P}(\Delta)$ for a binary power context family $\vec{\mathbb{K}}$ on a universe Δ with attribute sets $M_\mathcal{C}, M_\mathcal{R}$ is defined recursively:*

$$\begin{aligned} [\![\top]\!]_{\vec{\mathbb{K}}} &:= \Delta, \\ [\![\bot]\!]_{\vec{\mathbb{K}}} &:= \emptyset, \\ [\![m]\!]_{\vec{\mathbb{K}}} &:= m^{I_\mathcal{C}} \textit{ for } m \in M_\mathcal{C}, \\ [\![\varphi \sqcap \psi]\!]_{\vec{\mathbb{K}}} &:= [\![\varphi]\!]_{\vec{\mathbb{K}}} \cap [\![\psi]\!]_{\vec{\mathbb{K}}}, \\ [\![\exists r.\varphi]\!]_{\vec{\mathbb{K}}} &:= \{x \in \Delta \mid \exists y : (x,y) \in r^{I_\mathcal{R}} \wedge y \in [\![\varphi]\!]_{\vec{\mathbb{K}}}\} \textit{ for } r \in M_\mathcal{R}, \\ [\![\forall r.\varphi]\!]_{\vec{\mathbb{K}}} &:= \{x \in \Delta \mid \forall y : (x,y) \in r^{I_\mathcal{R}} \to y \in [\![\varphi]\!]_{\vec{\mathbb{K}}}\} \textit{ for } r \in M_\mathcal{R}. \end{aligned}$$

[2] Whenever in this article we use the term *concept* we refer to the notion used in Description Logic. If we want to refer to the meaning used in Formal Concept Analysis (FCA) we use *formal concept*.

Verbally, we have defined an extensional semantics, assigning to every concept description all entities of the universe fulfilling that description.

Furthermore, we say a formula φ is VALID IN $\overrightarrow{\mathbb{K}}$ (which we denote by $\overrightarrow{\mathbb{K}} \models \varphi$), iff $[\![\varphi]\!]_{\overrightarrow{\mathbb{K}}} = \Delta$. A formula φ ENTAILS a formula ψ IN $\overrightarrow{\mathbb{K}}$ (write: $\varphi \models_{\overrightarrow{\mathbb{K}}} \psi$), iff $[\![\varphi]\!]_{\overrightarrow{\mathbb{K}}} \subseteq [\![\psi]\!]_{\overrightarrow{\mathbb{K}}}$. A formula φ ENTAILS a formula ψ in general (write: $\varphi \models \psi$), iff $[\![\varphi]\!]_{\overrightarrow{\mathbb{K}}} \subseteq [\![\psi]\!]_{\overrightarrow{\mathbb{K}}}$ for all binary power context families $\overrightarrow{\mathbb{K}}$ with appropriate signature. Two formulae φ and ψ are called $\overrightarrow{\mathbb{K}}-$EQUIVALENT, iff $\varphi \models_{\overrightarrow{\mathbb{K}}} \psi$ and $\psi \models_{\overrightarrow{\mathbb{K}}} \varphi$ (write: $\varphi \equiv_{\overrightarrow{\mathbb{K}}} \psi$). They are EQUIVALENT, iff $\varphi \models \psi$ and $\psi \models \varphi$ (write: $\varphi \equiv \psi$).

Let $C = \{c_1, \ldots, c_n\}$ be a finite set of $\mathcal{FLE}$ concept descriptions. Then the new concept description $c_1 \sqcap \ldots \sqcap c_n$ will be abbreviated by $\bigsqcap C$. Moreover, let $\bigsqcap\{c\} = c$ and $\bigsqcap\emptyset = \top$.

Finally, note that $\forall r.(c_1 \sqcap c_2)$ and $\forall r.c_1 \sqcap \forall r.c_2$ are equivalent for any concept descriptions c_1, c_2 and any role r. So, for every concept description $c \in \mathcal{FLE}$ there is an equivalent "sibling" $\widetilde{c} \in \mathcal{FLE}$ where in no subformula $\forall r.\varphi$ the φ is a conjunction. In the sequel, we assume any formula we deal with to be normalized in this way.

The following abbreviations for sets of $\mathcal{FLE}$ concept descriptions have been found both intuitive and useful:

$$[\exists r]A := \begin{cases} \{\bot\} \text{ if } \bot \in A, \\ \{\exists r.\bigsqcap A\} \text{ otherwise} \end{cases}$$

and

$$[\forall r]A := \{\forall r.a \mid a \in A\}.$$

By restricting to a reduced amount of logic features (omitting disjunction and negation) we obtain a class of propositions that can be managed algorithmically in practice and - as we suppose (see also [15]) - still comprises the majority of human conceptual thinking.

4 $\mathcal{FLE}$-Contexts

As our aim is to use the exploration algorithm in order to collect information expressable by $\mathcal{FLE}$ statements it is just natural to define a kind of formal context, where the attributes are arbitrary $\mathcal{FLE}$ concept descriptions:

Definition 6. *Given a binary power context family $\overrightarrow{\mathbb{K}} = (\mathbb{K}_{\mathcal{C}}, \mathbb{K}_{\mathcal{R}})$ on a universe Δ and a set $M \subseteq \mathcal{FLE}(M_{\mathcal{C}}, M_{\mathcal{R}})$, the corresponding $\mathcal{FLE}$-CONTEXT is defined in the following way:*

$$\mathbb{K}_{\mathcal{FLE}}(M) := (\Delta, M, I) \text{ with } \delta I m :\Leftrightarrow \delta \in [\![m]\!]_{\overrightarrow{\mathbb{K}}}.$$

Formal contexts where attributes are DL-formulae and the incidence relation is defined via validity have been described by Prediger in [12]. While she was aiming at extending a context by "interesting" new attributes, we want to explore $\mathcal{FLE}$-contexts and therefore more attention at the choice of attributes is required as we will see in the sequel.

Now, what does validity of an implication in such an $\mathcal{FLE}$-context mean from the point of view of DL? Suppose

$$\{m_1, \ldots, m_k\} \to \{m_{k+1}, \ldots, m_l\}$$

is an implication valid in $\mathbb{K}_{\mathcal{FLE}}$. The following theorem shows that this is equivalent to the validity of the entailment statement

$$\bigsqcap\{m_1, \ldots, m_k\} \models_{\vec{\mathbb{K}}} \bigsqcap\{m_{k+1}, \ldots, m_l\}.$$

Theorem 1. *Let $\vec{\mathbb{K}}$ be a binary power context family, $M \subseteq \mathcal{FLE}$ and $A, B \subseteq M$. Then the implication $A \to B$ is valid in $\mathbb{K}_{\mathcal{FLE}}(M)$ iff $\bigsqcap A \models_{\vec{\mathbb{K}}} \bigsqcap B$.*

Proof. $\mathbb{K}_{\mathcal{FLE}}(M) \models A \to B$ iff for all $\delta \in \Delta$ from $A \subseteq \delta^I$ follows $B \subseteq \delta^I$. This is the case iff $\bigcap\{a^I \mid a \in A\} \subseteq \bigcap\{b^I \mid b \in B\}$, which due to the definition of I is equivalent to $\bigcap\{[\![a]\!]_{\vec{\mathbb{K}}} \mid a \in A\} \subseteq \bigcap\{[\![b]\!]_{\vec{\mathbb{K}}} \mid b \in B\}$ and thus also to $[\![\bigsqcap A]\!]_{\vec{\mathbb{K}}} \subseteq [\![\bigsqcap B]\!]_{\vec{\mathbb{K}}}$. $\square$

So this is the way how implications gained by an exploration process of an $\mathcal{FLE}$-context can be reinterpreted as entailment rules.

5 Successive Exploration

In the past there have been some approaches to apply the FCA exploration technique to a logic more expressive than propositional logic. Zickwolff used Ganter's algorithm to determine the first order Horn theory of a certain domain in [17].

In this section we describe the multistep exploration algorithm in detail.

At first, we have to stipulate $M_{\mathcal{C}}$ and $M_{\mathcal{R}}$ - the concepts and roles which's domain specific interrelationships we are interested in.

Next we may provide some empirical data by naming known entities $G \subseteq \Delta$ of the considered universe and stating their attributes.

Moreover, we can input axiomatic information about the domain by stating $\mathcal{FLE}$ entailment statements already known to hold.

So, we start the exploration sequence with the context $\mathbb{K}_0 = (G_0, M_0, I_0)$ where
$G_0 := G$,
$M_0 := M_{\mathcal{C}} \cup \{\perp\}$, and
$I_0 := I_{\mathcal{C}} \cap G \times M_{\mathcal{C}}$.

The exploration is carried out as described in Section 2. Every implication $\{m_1, \ldots, m_k\} \to \{m_{k+1}, \ldots, m_l\}$ being presented to the expert has to be interpreted in the following way: "Do all entities from the universe that fulfill the concept description $m_1 \sqcap \ldots \sqcap m_k$ also fulfill the concept description $m_{k+1} \sqcap \ldots \sqcap m_l$?" The expert either confirms this, or provides an entity that violates this condition. The result of this first exploration step is the stem base $\mathfrak{B}_0$.

When one such step (say: the one generating $\mathfrak{B}_{i-1}$) has finished, the next one has to be prepared:

First, the attribute set M_i is generated as follows:

$$\begin{aligned} M_i := M_{\mathcal{C}} \cup \{\bot\} \\ &\cup \{\exists r.\textstyle\bigsqcap A \mid r \in M_{\mathcal{R}}, A = A^{\mathfrak{B}_{i-1}} \subseteq M_{i-1} \setminus \{\bot\}\} \\ &\cup \{\forall r.m \mid r \in M_{\mathcal{R}}, m \in M_{i-1}\}. \end{aligned}$$

This choice of the attributes is motivated by the purpose to keep the set of attributes small (which is essential for the exploration algorithm since its worst case complexity increases exponentially with the number of attributes involved) while preserving the completeness we deal with in Section 6.

Note that for every attribute $m \in M_{i-1}$ we find an attribute $\widetilde{m} \in M_i$ with $m \equiv_{\mathbb{K}} \widetilde{m}$ by using the function $\varphi_i : \mathcal{FLE} \to \mathcal{P}(\mathcal{FLE})$ defined as follows:

$$\begin{aligned} \varphi_i(c) &:= \{c\} \text{ for } c \in M_{\mathcal{C}} \cup \{\bot\} \\ \varphi_i(\forall r.c) &:= [\forall r]\varphi_{i-1}(c) \\ \varphi_i(\exists r.c) &:= [\exists r](\varphi_{i-1}(c))^{\mathfrak{B}_{i-1}} \\ \varphi_i(\textstyle\bigsqcap C) &:= \bigcup\{\varphi_i(c) \mid c \in C\} \end{aligned}$$

It is easy to see that for $m \in M_{i-1}$ the term $\varphi_i(m)$ yields a singleton set. Now we take the only element of this set as our representative $\widetilde{m}$. The facts $\widetilde{m} \in M_i$ and $m \equiv_{\mathbb{K}} \widetilde{m}$ are immediate consequences of the following proofs.

Theorem 2. *Let $c \in \mathcal{FLE}_i$. Then $\varphi_i(c) \subseteq M_i$.*

Proof. We show this using induction on the role depth of c considering four cases:

- $c \in M_{\mathcal{C}} \cup \{\bot\}$. Then, by definition, $\{c\} \subseteq M_i$.
- $c = \forall r.\widetilde{c}$. As our induction hypothesis assures, we have $\varphi_{i-1}(\widetilde{c}) \subseteq M_{i-1}$ and due to the definition of M_i this directly implies $[\forall r]\varphi_{i-1}(\widetilde{c}) \subseteq M_i$.
- $c = \exists r.\widetilde{c}$. As induction hypothesis we have $\varphi_{i-1}(\widetilde{c}) \subseteq M_{i-1}$. But then we have also $\exists r.\bigsqcap\varphi(\widetilde{c}, i-1)^{\mathfrak{B}_{i-1}} \in M_i$, as a look at the constructive definition of M_i immediately shows.
- $c = \bigsqcap\widetilde{C}$. W.l.o.g. we presuppose there is no conjunction outside the quantifier range in any $\widetilde{c} \in \widetilde{C}$. So we have $\varphi_i(\widetilde{c}) \subseteq M_i$, due to the three cases above. □

Lemma 1. *For any $A \subseteq M_i$ we have $\bigsqcap A \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap A^{\mathfrak{B}_i}$.*

Proof. From Theorem 1 we know that for every entity $\delta \in \Delta$ the set of its attributes $M \in \mathcal{FLE}_i$ fulfills all implications from $\mathfrak{B}_i$. Hence, when considering only those δ having all attributes from A, every one of them must even have every attribute from $A^{\mathfrak{B}_i}$, since this is the smallest attribute set containing A and satisfying $\mathfrak{B}_i$. Therefore we have $\bigcap\{[\![m]\!]_{\overrightarrow{\mathbb{K}}} \mid m \in A\} \subseteq \bigcap\{[\![m]\!]_{\overrightarrow{\mathbb{K}}} \mid m \in A^{\mathfrak{B}_i}\}$.

On the other hand we have trivially $\bigcap\{[\![m]\!]_{\overrightarrow{\mathbb{K}}} \mid m \in A^{\mathfrak{B}_i}\} \subseteq \bigcap\{[\![m]\!]_{\overrightarrow{\mathbb{K}}} \mid m \in A\}$, for the left hand side intersection contains at least all sets from the right hand side intersection. So, finally we get

$$[\![\bigsqcap A]\!]_{\overrightarrow{\mathbb{K}}} = \bigcap\{[\![m]\!]_{\overrightarrow{\mathbb{K}}} \mid m \in A\} = \bigcap\{[\![m]\!]_{\overrightarrow{\mathbb{K}}} \mid m \in A^{\mathfrak{B}_i}\} = [\![\bigsqcap A^{\mathfrak{B}_i}]\!]_{\overrightarrow{\mathbb{K}}}.$$

□

Theorem 3. *Let $c \in \mathcal{FLE}_i$. Then $c \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap \varphi_i(c)$.*

Proof. We show this again via induction on the role depth:

- $c \in M_{\mathcal{C}} \cup \{\bot\}$. Then we have $[\![c]\!]_{\overrightarrow{\mathbb{K}}} = [\![\bigsqcap\{c\}]\!]_{\overrightarrow{\mathbb{K}}}$.
- $c = \forall r.\widetilde{c}$. By induction hypothesis we have $[\![\widetilde{c}]\!]_{\overrightarrow{\mathbb{K}}} = [\![\bigsqcap \varphi_{i-1}(\widetilde{c})]\!]_{\overrightarrow{\mathbb{K}}}$ implying $[\![\forall r.\widetilde{c}]\!]_{\overrightarrow{\mathbb{K}}} = [\![\bigsqcap [\forall r] \varphi_{i-1}(\widetilde{c})]\!]_{\overrightarrow{\mathbb{K}}}$ which by definition equals $[\![\bigsqcap \varphi_i(\forall r.\widetilde{c})]\!]_{\overrightarrow{\mathbb{K}}}$.
- $c = \exists r.\widetilde{c}$. By induction hypothesis we have $[\![\widetilde{c}]\!]_{\overrightarrow{\mathbb{K}}} = [\![\bigsqcap \varphi_{i-1}(\widetilde{c})]\!]_{\overrightarrow{\mathbb{K}}}$, and since $[\![\bigsqcap \varphi_{i-1}(\widetilde{c})]\!]_{\overrightarrow{\mathbb{K}}} = [\![\bigsqcap \varphi_{i-1}(\widetilde{c})^{\mathfrak{B}_{i-1}}]\!]_{\overrightarrow{\mathbb{K}}}$ due to Lemma 1 we have $[\![\exists r.\widetilde{c}]\!]_{\overrightarrow{\mathbb{K}}} = [\![\exists r.\bigsqcap(\varphi_{i-1}(\widetilde{c}))^{\mathfrak{B}_{i-1}}]\!]_{\overrightarrow{\mathbb{K}}}$ which by definition equals $[\![\bigsqcap \varphi_i(\exists r.\widetilde{c})]\!]_{\overrightarrow{\mathbb{K}}}$.
- $c = \bigsqcap \widetilde{C}$. Again we can preassume no conjunction outside the quantifier range in any $\widetilde{c} \in \widetilde{C}$. Then $[\![\bigsqcap \widetilde{C}]\!]_{\overrightarrow{\mathbb{K}}} = \bigcap\{[\![\widetilde{c}]\!]_{\overrightarrow{\mathbb{K}}} \mid \widetilde{c} \in \widetilde{C}\} = \bigcap\{[\![\bigsqcap \varphi_i(\widetilde{c})]\!]_{\overrightarrow{\mathbb{K}}} \mid \widetilde{c} \in \widetilde{C}\}$ because of the cases shown before. Now, this is obviously the same as $\bigcap\{[\![m]\!]_{\overrightarrow{\mathbb{K}}} \mid m \in \varphi_i(\widetilde{c}), \widetilde{c} \in \widetilde{C}\} = [\![\bigsqcap(\bigcup\{\varphi_i(\widetilde{c}) \mid \widetilde{c} \in \widetilde{C}\})]\!]_{\overrightarrow{\mathbb{K}}}$. □

This allows us to reuse all implications from the former exploration step as input for the next one: We simply add $\varphi_i(\bigsqcap A) \rightarrow \varphi_i(\bigsqcap B)$ for all $A \rightarrow B \in \mathfrak{B}_{i-1}$ to the background knowledge.

But there is more a priori knowledge that can be extracted from $\mathfrak{B}_{i-1}$. Exploiting the deduction calculus presented in the appendix, we can augment our a priori information even further. So we add:

- $\{\bot\} \rightarrow M_i$ (due to the $\mathcal{C}$ rule),

– $\{\exists r.\bigsqcap A\} \rightarrow \{\exists r.\bigsqcap B\}$ for all $B \subseteq A$ (as a consequence of the $\mathcal{ID}$, $\mathcal{PE}$, and $\exists\mathcal{L}$ rules),
– $\{\exists r.\bigsqcap A, \forall r.b\} \rightarrow \{\exists r.\bigsqcap (A \cup \{b\})^{\mathfrak{B}_i}\}$ (because of the rules $\forall\mathcal{P}$ and $\exists\mathcal{L}$),
– $\{\forall r.a \mid a \in A\} \rightarrow \{\forall r.b \mid b \in B\}$ for all $A \rightarrow B \in \mathfrak{B}_i$ (justified by $\forall\mathcal{L}$).[3]

This algorithm can be carried out iteratively, thereby producing a sequence of implication bases $\mathfrak{B}_0, \mathfrak{B}_1 \dots$. How these can be used for deciding "entailment queries" will be dealt with in the next chapter.

Baader presented a method for computing the subsumption hierarchy of all concept descriptions, that can be obtained by applying conjunction to concept names in [1]. His algorithm technically corresponds to our first exploration step (on the attribute set M_0) - but for the intended purpose: Baader suggests to let a DL subsumption algorithm take the role of the expert thus exploring the subsumptions valid for a given DL system, while we are aiming at finding information not yet being inherently present in the system.

6 Checking the Validity of an Entailment Statement

This section is dedicated to the question, which kind of information will be acquired after a certain step of the exploration algorithm. The answer is the following. Having explored a binary power context family until step i, we can decide for *any* entailment statement $c_1 \models_{\vec{\mathbb{K}}} c_2$ (with c_1, c_2 being arbitrary $\mathcal{FLE}$ concept descriptions with maximal role depth of at most i) whether it is valid in $\vec{\mathbb{K}}$ or not, using just the bases $\mathfrak{B}_0, \dots, \mathfrak{B}_i$. In this sense the exploration algorithm is complete. The decision procedure works as follows: The function φ_i, defined in the preceding section, gives for any $c \in \mathcal{FLE}_i$ a set of attributes from M_i the conjunction of which yields a concept description that is $\vec{\mathbb{K}}$-equivalent to c. Therefore, as the following corollary shows the entailment of two concept descriptions c_1 and c_2 can simply be checked by exploiting the proposition

$$c_1 \models_{\vec{\mathbb{K}}} c_2 \iff \varphi_i(c_2)^{\mathfrak{B}_i} \subseteq \varphi_i(c_1)^{\mathfrak{B}_i}.$$

Corollary 1. *Let $c_1, c_2 \in \mathcal{FLE}_i$. Then $c_1 \models_{\vec{\mathbb{K}}} c_2$ iff $\varphi_i(c_2)^{\mathfrak{B}_i} \subseteq \varphi_i(c_1)^{\mathfrak{B}_i}$.*

Proof. Due to Theorem 3, $c_1 \models_{\vec{\mathbb{K}}} c_2$ is equivalent to $\bigsqcap\varphi_i(c_1) \models_{\vec{\mathbb{K}}} \bigsqcap\varphi_i(c_2)$. According to Theorem 2, we have $\varphi_i(c_1) \subseteq M_i$ and $\varphi_i(c_2) \subseteq M_i$. So via Theorem 1, this means the same as the validity of the implication $\varphi_i(c_1) \rightarrow \varphi_i(c_2)$ in $\mathbb{K}_i$. Obviously, this is equivalent to $\varphi_i(c_2)^{\mathfrak{B}_i} \subseteq \varphi_i(c_1)^{\mathfrak{B}_i}$. □

By checking entailment in both directions we also may find equivalences in $\vec{\mathbb{K}}$.

[3] Mark that the $\exists\mathcal{F}$-rule is already incorporated in the algorithm via the definition of $[\exists r]$. The $\mathcal{ID}$, $\mathcal{PE}$ and $\mathcal{MP}$ rules (being the usual implication deduction or Armstrong rules) do not need to be cared about since we are looking for implication bases, being just sets reduced with respect to the Armstrong rules.

7 Termination

At least from the theoretical point of view the question emerges, whether and under which circumstances the proposed algorithm terminates, i.e., all information expressable by $\mathcal{FLE}$ entailment statements has been acquired. We found that this is the case iff there is an $n \in \mathbb{N}$ such that the mapping $F_n : \{A^{\mathfrak{B}_n} \mid A \subseteq M_n\} \to \{B^{\mathfrak{B}_{n+1}} \mid B \subseteq M_{n+1}\}$ with $F_n(A) := (\varphi_{n+1}(\bigsqcap A))^{\mathfrak{B}_{n+1}}$ is a bijection between the $\mathfrak{B}_n$-closed subsets from M_n and the $\mathfrak{B}_{n+1}$-closed subsets from M_{n+1}.

In the the following theorems we show some consequences of this property:

Theorem 4. *Let $\vec{\mathbb{K}}$ be a binary power context family with the property described above. Then*

1. *For any $B = B^{\mathfrak{B}_{n+1}} \subseteq M_{n+1}$ and $A = F_n^{-1}(B)$ we have $\bigsqcap A \equiv_{\vec{\mathbb{K}}} \bigsqcap B$.*
2. *For any $c \in \mathcal{FLE}_{n+1}$ we have $c \equiv_{\vec{\mathbb{K}}} \bigsqcap F_n^{-1}(\varphi_{n+1}(c))^{\mathfrak{B}_{n+1}}$.*

Proof.

Because $A \subseteq M_n$ we know $\bigsqcap A \equiv_{\vec{\mathbb{K}}} \bigsqcap \varphi_{n+1}(\bigsqcap A)$ due to Theorem 3. Applying Lemma 1 gives us $\bigsqcap \varphi_{n+1}(\bigsqcap A) \equiv_{\vec{\mathbb{K}}} \bigsqcap(\varphi_{n+1}(\bigsqcap A))^{\mathfrak{B}_{n+1}}$. By definition of F_n we see that the right hand side of the equivalence is just $\bigsqcap F_n(A)$. Since $F_n(A) = B$, we are done.

The second proposition can then be proved as follows. We know $c \equiv_{\vec{\mathbb{K}}} \bigsqcap \varphi_{n+1}(c)$ by Theorem 3 and $\bigsqcap \varphi_{n+1}(c) \equiv_{\vec{\mathbb{K}}} \bigsqcap(\varphi_{n+1}(c))^{\mathfrak{B}_{n+1}}$ by Lemma 1. From the first part of this theorem follows $\bigsqcap(\varphi_{n+1}(c))^{\mathfrak{B}_{n+1}} \equiv_{\vec{\mathbb{K}}} \bigsqcap F_n^{-1}(\varphi_{n+1}(c))^{\mathfrak{B}_{n+1}}$. □

This theorem provides a way to "shrink" an $\mathcal{FLE}_{n+1}$ formula to maximal quantifier depth n preserving its semantics. But - exploiting this fact - we can do even more: for any concept description $c \in \mathcal{FLE}$ we find an empirically equivalent concept description $\widetilde{c} \in \mathcal{FLE}_n$ by applying the function $\pi : \mathcal{FLE} \to \mathcal{FLE}_n$ with[4]

$$
\begin{aligned}
& d \mapsto d \text{ for all } d \in M_{\mathcal{C}} \cup \{\bot\} \\
& \exists r.d \mapsto \begin{cases} \bigsqcap[\exists r](\varphi_{n-1}(d))^{\mathfrak{B}_{n-1}} \text{ if } \exists r.d \in \mathcal{FLE}_n, \\ \bigsqcap F_n^{-1}([\exists r](\varphi_n(\pi(d)))^{\mathfrak{B}_n})^{\mathfrak{B}_{n+1}} \text{ otherwise.} \end{cases} \\
& \forall r.d \mapsto \begin{cases} \bigsqcap[\forall r]\varphi_{n-1}(d) \text{ if } \forall r.d \in \mathcal{FLE}_n, \\ \bigsqcap F_n^{-1}([\forall r]\varphi_n(\pi(d)))^{\mathfrak{B}_{n+1}} \text{ otherwise.} \end{cases} \\
& \bigsqcap D \mapsto \bigsqcap\{\pi(d) \mid d \in D\}.
\end{aligned}
$$

[4] In this notation, $(.)^{\mathfrak{B}}$ binds stronger than F_n^{-1}, φ_n, φ_{n-1}, $[\forall r]$, and $[\exists r]$.

Theorem 5. *Let $\overrightarrow{\mathbb{K}}$ be a binary power context family, $n \in \mathbb{N}$, and the corresponding F_n be a bijection. Then for any $c \in \mathcal{FLE}$ we have $\pi(c) \in \mathcal{FLE}_n$ and $\pi(c) \equiv_{\overrightarrow{\mathbb{K}}} c$.*

Proof. This proof will be done by induction on the quantifier depth. We have to consider the following cases:

- $c \in M_{\mathcal{C}} \cup \{\perp\}$.
  This is trivial: $c \equiv_{\overrightarrow{\mathbb{K}}} c = \pi(c)$.
- $c = \exists r.d \in \mathcal{FLE}_n$.
  Applying Theorem 3 and Lemma 1 yields $d \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap \varphi_{n-1}(d) \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap(\varphi_{n-1}(d))^{\mathfrak{B}_{n-1}}$, directly implying $\exists r.d \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap[\exists r](\varphi_{n-1}(d))^{\mathfrak{B}_{n-1}} = \pi(c)$. Since $\varphi_{n-1}(d) \subseteq \mathcal{FLE}_{n-1}$, we also have $\pi(c) \subseteq \mathcal{FLE}_n$.
- $c = \exists r.d \notin \mathcal{FLE}_n$.
  By induction hypothesis, $d \equiv_{\overrightarrow{\mathbb{K}}} \pi(d)$. Theorem 3 and Lemma 1 give us $\pi(d) \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap(\varphi_n(\pi(d)))^{\mathfrak{B}_n}$. From this we conclude $\exists r.d \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap[\exists r](\varphi_n(\pi(d)))^{\mathfrak{B}_n}$. Notice that the equivalence's right hand side is in M_{n+1} assured by Theorem 2 and the Definition of M_{n+1}. Applying Lemma 1 once more we get $\bigsqcap[\exists r](\varphi_n(\pi(d)))^{\mathfrak{B}_n} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap([\exists r](\varphi_n(\pi(d)))^{\mathfrak{B}_n})^{\mathfrak{B}_{n+1}}$ and by Theorem 4.1 we have $\bigsqcap([\exists r](\varphi_n(\pi(d)))^{\mathfrak{B}_n})^{\mathfrak{B}_{n+1}} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap F_n^{-1}([\exists r](\varphi_n(\pi(d)))^{\mathfrak{B}_n})^{\mathfrak{B}_{n+1}} = \pi(c)$. So we have showed $c \equiv_{\overrightarrow{\mathbb{K}}} \pi(c)$. The application of F_n^{-1} assures $\pi(c) \in \mathcal{FLE}_n$.
- $c = \forall r.d \in \mathcal{FLE}_n$.
  Applying Theorem 3 yields $d \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap \varphi_{n-1}(d)$, directly implying $\forall r.d \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap[\forall r]\varphi_{n-1}(d) = \pi(c)$. Since $\varphi_{n-1}(d) \subseteq \mathcal{FLE}_{n-1}$, we also have $\pi(c) \subseteq \mathcal{FLE}_n$.
- $c = \forall r.d \notin \mathcal{FLE}_n$.
  By induction hypothesis, $d \equiv_{\overrightarrow{\mathbb{K}}} \pi(d)$. Theorem 3 gives us $\pi(d) \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap \varphi_n(\pi(d))$. From this we conclude $\forall r.d \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap[\forall r]\varphi_n(\pi(d))$. Notice that $[\forall r]\varphi_n(\pi(d)) \subseteq M_{n+1}$ assured by Theorem 2 and the Definition of M_{n+1}. Applying Lemma 1 we get $\bigsqcap[\forall r]\varphi_n(\pi(d)) \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap([\forall r]\varphi_n(\pi(d)))^{\mathfrak{B}_{n+1}}$ and by the first proposition of theorem 4 we have $\bigsqcap([\forall r]\varphi_n(\pi(d)))^{\mathfrak{B}_{n+1}} \equiv_{\overrightarrow{\mathbb{K}}} \bigsqcap F_n^{-1}([\forall r]\varphi_n(\pi(d)))^{\mathfrak{B}_{n+1}} = \pi(c)$. So we have showed $c \equiv_{\overrightarrow{\mathbb{K}}} \pi(c)$. The application of F_n^{-1} assures $\pi(c) \in \mathcal{FLE}_n$.
- $c = \bigsqcap D$.
  W.l.o.g. we can assume, that every $d \in D$ has no conjunction outside the range of a quantifier, thus, one of the cases above is applicable. Therefore,

we know $\pi(d) \in \mathcal{FLE}_n$ and $d \equiv_{\vec{\mathbb{K}}} \pi(d)$ for every $d \in D$. This implies $\bigsqcap D \equiv_{\vec{\mathbb{K}}} \bigsqcap\{\pi(d) \mid d \in D\} = \pi(c)$ as well as $\pi(c) \in \mathcal{FLE}_n$.

□

In words, the π function just realizes the following transformation: beginning from "inside" the concept expression c, subformulae having maximal role depth of $n+1$ are substituted by equivalent ones with smaller role depth. When applied iteratively, this results in a formula $\widetilde{c}$ from $\mathcal{FLE}_n$ that is equivalent to the original one. This formula's validity can now be checked by the method described in the preceding section.

It is easy to show, that the termination criterion mentioned above is equivalent to the finiteness of $\mathcal{FLE}/\equiv_{\vec{\mathbb{K}}}$, which is trivially fulfilled if Δ is finite.

8 A Small Example

After having presented the algorithm in theory, we will consider an easy example for our method in order to show what type of information we can expect from this method. Let the universe Δ be the natural numbers including zero. Furthermore, let $M_\mathcal{C}$ and $M_\mathcal{R}$ be defined as shown in Figure 2 a). Carrying out the exploration on $\mathbb{K}_0$ (where the attributes M_0 are just the elements from $M_\mathcal{C}$ plus $\perp$) we get the implication base $\mathfrak{B}_0$ shown in Figure 2 b).

After this step, we generate the attribute set M_1 for the next one. First we reuse all attributes from M_0, second we take the conjunction over any $\mathfrak{B}_0$-closed subset not containing $\perp$ of M_0 preceded by an existential quantifier, and third we include all combinations of a universal quantifier and one attribute from M_0. Figure 3 lists the attributes from M_1.

$c \in M_\mathcal{C}$	name	$c^{I_\mathcal{C}}$
ev	even	$\{2n \mid n \in \mathbb{N}\}$
od	odd	$\{2n+1 \mid n \in \mathbb{N}\}$
pr	prime	$\{n \geq 2 \mid kl = n \Rightarrow k \in \{1, n\}\}$
$e0$	equals zero	$\{0\}$
$e1$	equals one	$\{1\}$
$e2$	equals two	$\{2\}$
$g2$	greater than two	$\{n \in \mathbb{N} \mid n \geq 3\}$

$r \in M_\mathcal{R}$	name	$r^{I_\mathcal{R}}$
s	successor	$\{(n, n+1) \mid n \in \mathbb{N}\}$
p	predecessor	$\{(n+1, n) \mid n \in \mathbb{N}\}$
d	divisor	$\{(m, n) \mid \exists k \in \mathbb{N} : m = kn\}$
m	multiple	$\{(n, m) \mid \exists k \in \mathbb{N} : m = kn\}$

$\{e0\}$	$\to \{ev\}$
$\{e1\}$	$\to \{od\}$
$\{e2\}$	$\to \{ev, pr\}$
$\{ev, pr\}$	$\to \{e2\}$
$\{od, pr\}$	$\to \{g2\}$
$\{pr, g2\}$	$\to \{od\}$
$\{ev, od\}$	$\to \{\perp\}$
$\{g2, e0\}$	$\to \{\perp\}$
$\{g2, e1\}$	$\to \{\perp\}$
$\{e0, e2\}$	$\to \{\perp\}$

Fig. 2. Attributes $M_\mathcal{C}$, $M_\mathcal{R}$ and definition of the incidence relations $I_\mathcal{C}$, $I_\mathcal{R}$ for the example and the implication base $\mathfrak{B}_0$ resulting from the first exploration step.

$g2$	pr	od	ev	$e1$	$e0$	$e2$	$\bot$			
$\exists s.\top$	$\exists s.g2$	$\exists s.pr$	$\exists s.od$	$\exists s.ev$	$\exists s.(od \sqcap g2)$	$\exists s.(od \sqcap e1)$	$\exists s.(ev \sqcap g2)$	$\exists s.(ev \sqcap e0)$	$\exists s.(od \sqcap g2 \sqcap pr)$	$\exists s.(ev \sqcap pr \sqcap e2)$
$\exists p.\top$	$\exists p.g2$	$\exists p.pr$	$\exists p.od$	$\exists p.ev$	$\exists p.(od \sqcap g2)$	$\exists p.(od \sqcap e1)$	$\exists p.(ev \sqcap g2)$	$\exists p.(ev \sqcap e0)$	$\exists p.(od \sqcap g2 \sqcap pr)$	$\exists p.(ev \sqcap pr \sqcap e2)$
$\exists m.\top$	$\exists m.g2$	$\exists m.pr$	$\exists m.od$	$\exists m.ev$	$\exists m.(od \sqcap g2)$	$\exists m.(od \sqcap e1)$	$\exists m.(ev \sqcap g2)$	$\exists m.(ev \sqcap e0)$	$\exists m.(od \sqcap g2 \sqcap pr)$	$\exists m.(ev \sqcap pr \sqcap e2)$
$\exists d.\top$	$\exists d.g2$	$\exists d.pr$	$\exists d.od$	$\exists d.ev$	$\exists d.(od \sqcap g2)$	$\exists d.(od \sqcap e1)$	$\exists d.(ev \sqcap g2)$	$\exists d.(ev \sqcap e0)$	$\exists d.(od \sqcap g2 \sqcap pr)$	$\exists d.(ev \sqcap pr \sqcap e2)$
$\forall s.g2$	$\forall s.pr$	$\forall s.od$	$\forall s.ev$	$\forall s.e1$	$\forall s.e0$	$\forall s.e2$	$\forall s.\bot$			
$\forall p.g2$	$\forall p.pr$	$\forall p.od$	$\forall p.ev$	$\forall p.e1$	$\forall p.e0$	$\forall p.e2$	$\forall p.\bot$			
$\forall m.g2$	$\forall m.pr$	$\forall m.od$	$\forall m.ev$	$\forall m.e1$	$\forall m.e0$	$\forall m.e2$	$\forall m.\bot$			
$\forall d.g2$	$\forall d.pr$	$\forall d.od$	$\forall d.ev$	$\forall d.e1$	$\forall d.e0$	$\forall d.e2$	$\forall d.\bot$			

Fig. 3. Attributes M_1 for the second exploration step.

Then we have to generate the a priori knowledge for the second exploration step. First, we use the information collected so far. When we proceed from the first ($i = 0$) to the second ($i = 1$) step we simply can use $\mathfrak{B}_0$ as additional a priori information without further adaption. Furthermore, applying the deduction consequences mentioned in Section 5 we have to add e.g.:

- $\{\bot\} \rightarrow M_1$
- $\{\exists s.(od \sqcap g2 \sqcap pr)\} \rightarrow \{\exists s.(od \sqcap g2)\}$
- $\{\exists s.pr, \forall s.g2\} \rightarrow \{\exists s.(od \sqcap g2 \sqcap pr)\}$
- $\{\forall p.ev, \forall p.od\} \rightarrow \{\forall p.e2\}$

After these preparations, the next exploration step is invoked. We visualize its result by the concept lattice in Figure 4, which mirrors the conceptual hierarchy of the formulae from M_1.

As an example, we will now demonstrate how to check the validity of the $\mathcal{FLE}_1$ entailment statement

$$pr \sqcap \exists s.(od \sqcap pr) \sqsubseteq_{\mathbb{K}} e2,$$

verbally: "is two the only prime having an odd prime sucessor?" Now, we carry out the necessary calculations and find

$$\begin{aligned}
&\varphi_1(pr \sqcap \exists s.(od \sqcap pr)) \\
&= \varphi_1(pr) \cup \varphi_1(\exists s.(od \sqcap pr)) \\
&= \varphi_1(pr) \cup [\exists r](\varphi_0(od \sqcap pr))^{\mathfrak{B}_0} \\
&= \varphi_1(pr) \cup [\exists r](\varphi_0(od) \cup \varphi_0(pr))^{\mathfrak{B}_0} \\
&= \{pr\} \cup [\exists r]\{od, pr\}^{\mathfrak{B}_0} \\
&= \{pr\} \cup [\exists r]\{od, pr, g2\} \\
&= \{pr, \exists r.(od \sqcap pr \sqcap g2)\}
\end{aligned}$$

as well as

$$\varphi_1(ev) = \{ev\}.$$

When applying the $\mathfrak{B}_1$-closure to both sets (the result is to large to be displayed here but can be derived from the line diagram next page) we find the outcomes

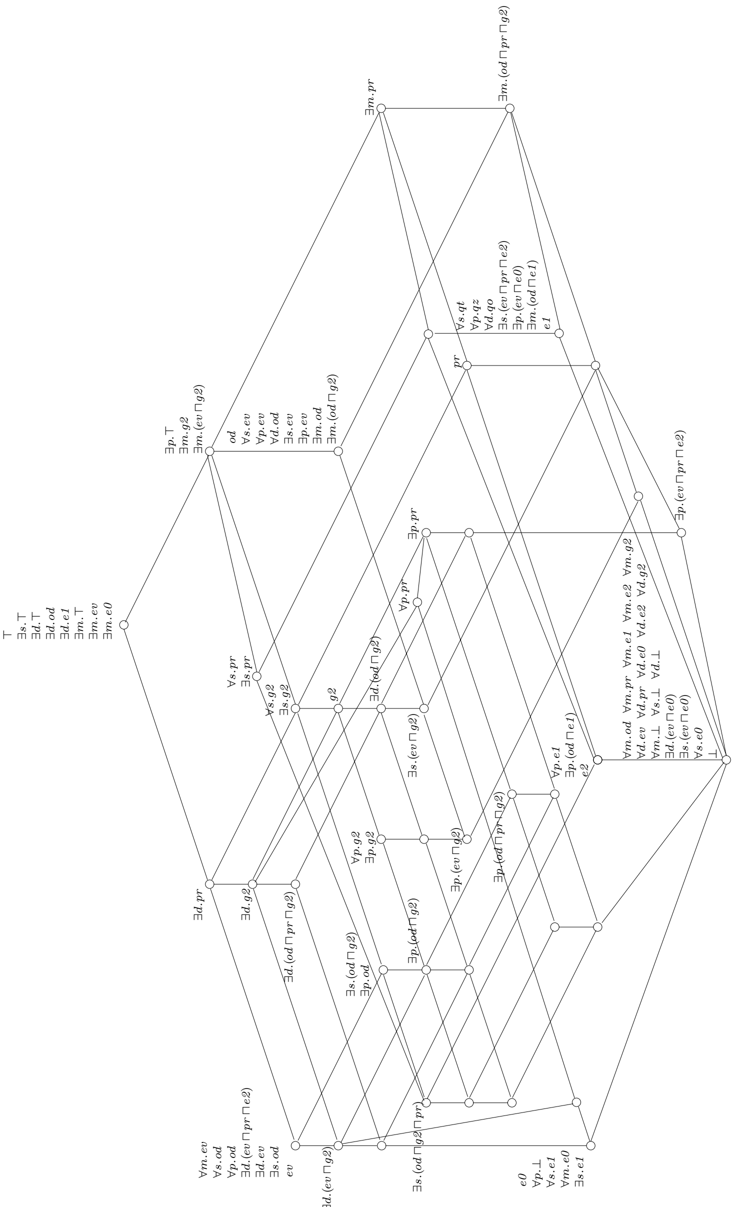

Fig. 4. Concept lattice from the second exploration step representing the implicational knowledge in $\mathbb{K}_1$.

even identical. Thus, the validity of our hypothetical entailment statement can be confirmed.

Finally we deal with the question whether the exploration algorithm terminates in our case after some step. This has to be denied for the following reason. Consider the infinite sequence *e0*, $\exists p.e0$, $\exists p.\exists p.e0, \ldots$. Every formula in this sequence is satisfied by exactly one natural number. Moreover, these numbers are all pairwise different. Therefore, every formula of the sequence is in another $\equiv_{\mathbb{K}}$-equivalence class, thus $\mathcal{FLE}/\equiv_{\mathbb{K}}$ is infinite in our case. Hence, the algorithm does not terminate.

9 A Possible Application: Ontology Exploration

After having dealt with details and theoretic properties of the algorithm as well as a small example we will now widen our scope and look for promising applications. As already said, we think the proposed algorithm could be very helpful in designing conceptual descriptions of world aspects. Markup ontologies are a very popular example for this. Although most of those descriptions are formulated in logics much more complex than $\mathcal{FLE}$ our algorithm is still applicable as long as there are complete reasoning algorithms for deciding subsumption and they contain $\mathcal{FLE}$ (for instance we have the FACT algorithm for reasoning in $\mathcal{SHIQ}(d)$ - see [10] and [9]). After stipulating the names and definitions of concepts and roles (and thereby specifying the fragment of reality to describe) the next step in designing an ontology would be to define axioms or rules stating how the specified concepts are interrelated. Our exploration algorithm can support this tedious and error-prone task by guiding the expert. Every potential axiom the algorithm comes up with will first be passed to the reasoning algorithm appropriate for the used logic. If this axiom can be proven it will be confirmed to the algorithm, if not the human expert has to be asked. If he judges the rule to be generally valid in the domain, a genuinely new axiom has been found and can be incorporated into the domain description. Otherwise the expert has to enter a counterexample, which violates the hypothetical axiom. One advantage of applying this technique is the guarantee, that all axioms expressable as $\mathcal{FLE}$ entailment statements with a certain role depth will certainly be found.

Finally we want to reply to a possible remark from the point of view of DL: one could object, that sometimes or even most times ontologies are designed for several different domains, such that an expert would not want to commit himself to one specific domain, as it is necessary when applying this algorithm. However, from the mathematical point of view this is not a severe problem: we just take the disjoint union of all domains we want to describe as reference domain of our exploration. A rule would be valid in this "superdomain" if and only if it is valid in all of the original domains.

For this reason we are very confident that an implementation of this algorithm could be a very helpful tool in order to build and refine domain descriptions - not only for working with ontologies. As there is a strong relationship between

DL and modal logic (which in turn can be enriched by temporal and epistemic features), applications for describing discrete dynamic systems and multi agent systems are in the realms of possibility.

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Appendix: A Deduction Calculus

Arising naturally from the considerations in this paper is the question for a deduction calculus for implications on $\mathcal{FLE}$. We found a set of deduction rules which we proved to be sound and complete. The completeness proof is based on a fixpoint construction of a standard model and beyond the scope and spatial capacity of this article. So we just present the deduction rules here.

Definition 7. *For given $M_{\mathcal{C}}, M_{\mathcal{R}}$ we define a ternary relation $Y \subseteq \mathcal{P}(\mathcal{FLE})^3$ as the smallest relation fulfilling the following conditions:*

$$
\begin{aligned}
&(\emptyset, \{\bot\}, \emptyset) \in Y,\\
&(\Phi, \Phi_1, \Phi_2) \in Y \Rightarrow ([\exists r]\Phi, [\exists r]\Phi_1, [\exists r]\Phi_2) \in Y,\\
&(\Phi, \Phi_1, \Phi_2) \in Y \Rightarrow ([\forall r]\Phi, [\forall r]\Phi_1, [\exists r]\Phi_2) \in Y,\\
&(\Phi, \Phi_1, \Phi_2) \in Y \Rightarrow (\Psi \cup \Phi, \Psi \cup \Phi_1, \Psi \cup \Phi_2) \in Y,\\
&(\Phi, \Phi_1, \Phi_2) \in Y \Rightarrow (\Phi, \Phi_2, \Phi_1) \in Y.
\end{aligned}
$$

The motivation for this definition is - roughly spoken - to encode case distinction. Note that, for all $(\Phi, \Phi_1, \Phi_2) \in Y$ and any binary power context family $\overrightarrow{\mathbb{K}}$, we have

$$\delta \in [\![\textstyle\bigsqcap\Phi]\!]_{\overrightarrow{\mathbb{K}}} \Leftrightarrow \delta \in [\![\textstyle\bigsqcap\Phi_1]\!]_{\overrightarrow{\mathbb{K}}} \vee \delta \in [\![\textstyle\bigsqcap\Phi_2]\!]_{\overrightarrow{\mathbb{K}}}$$

for all $\delta \in \Delta$, verbally: every entity of the universe fulfilling all descriptions from Φ fulfills all descriptions from Φ_1 or all descriptions from Φ_2.

Definition 8. *The set $\mathcal{DR}$ of* DERIVATION RULES *consists of the following rules ($a, b, c \in \mathcal{FLE}$, $A, B_1, \ldots, B_n, C, D_1, \ldots, D_k \in \mathcal{P}_{fin}(\mathcal{FLE})$, and $(\Phi, \Phi_1, \Phi_2) \in Y$)*

$$\frac{}{\bot \to \{a\}}\mathcal{C} \qquad \frac{A \to B}{[\exists r]A \to [\exists r]B}\exists\mathcal{L}$$

$$\frac{}{\{\exists r.\bot\} \to \{\bot\}}\exists\mathcal{F} \qquad \frac{}{[\exists r]A \cup \{\forall r.b\} \to [\exists r](A \cup \{b\})}\forall\mathcal{P}$$

$$\frac{}{A \to A}\mathcal{ID} \qquad \frac{A \to B}{[\forall r]A \to [\forall r]B}\forall\mathcal{L}$$

$$\frac{A \to B}{A \cup \{c\} \to B}\mathcal{PE} \qquad \frac{\Phi_1 \to A, \Phi_2 \to A}{\Phi \to A}\mathcal{CD}$$

$$\frac{A \to B, A \cup B \to C}{A \to C}\mathcal{MP}$$