ExpExpExplosion: Uniform Interpolation in General $\mathcal{EL}$ Terminologies

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Abstract. Although $\mathcal{EL}$ is a popular logic used in large existing knowledge bases, to the best of our knowledge no procedure has yet been proposed that computes uniform $\mathcal{EL}$ interpolants of general $\mathcal{EL}$ terminologies. Up to now, also the bounds on the size of uniform $\mathcal{EL}$ interpolants remain unknown. In this paper, we propose an approach based on proof theory and the theory of formal tree languages to computing a finite uniform interpolant for a general $\mathcal{EL}$ terminology if it exists. Further, we show that, if such a finite uniform $\mathcal{EL}$ interpolant exists, then there exists one that is at most triple exponential in the size of the original TBox, and that, in the worst-case, no shorter interpolants exist, thereby establishing the triple exponential tight bounds on their size.

1 Introduction

With the wide-spread adoption of ontological modeling by means of the W3C-specified OWL Web Ontology Language [15], description logics [2, 16] have developed into one of the most popular family of formalisms employed for knowledge representation and reasoning.

For application scenarios where scalability of reasoning is of utmost importance, specific tractable sublanguages (the so-called *profiles* [12]) of OWL have been put into place, among them OWL EL which in turn is based on DLs of the $\mathcal{EL}$ family [3, 1].

In view of this practical deployment of OWL and its profiles, the importance of non-standard reasoning services for supporting knowledge engineers in modeling a particular domain or in understanding existing models by visualizing implicit dependencies between concepts and roles was pointed out by the research community [4, 14]. An example of such reasoning services supporting knowledge engineers in different activities is that of *uniform interpolation*: given a theory using a certain vocabulary, and a subset of "relevant terms" of that vocabulary, find a theory that uses only the relevant terms and gives rise to the same consequences (expressible via relevant terms) as the original theory. In particular for the understanding and the development of complex knowledge bases, e.g., those consisting of *general concept inclusions* (GCIs), the appropriate tool support would be beneficial.

In our paper, we consider the task of uniform interpolation in the very lightweight description logic $\mathcal{EL}$. An existing approach [7] to uniform interpolation in $\mathcal{EL}$ is restricted to terminologies containing each atomic concept at most once on the left-hand side of concept inclusions and additionally satisfying sufficient, but not necessary acyclicity conditions. Lutz and Wolter [11] propose an approach to uniform interpolation in expressive description logics such as $\mathcal{ALC}$ featuring general terminologies, which, however does not solve the problem of uniform interpolation in $\mathcal{EL}$. Recently, Lutz, Seylan and Wolter [9] proposed an ExpTime procedure for deciding, whether a finite uniform $\mathcal{EL}$ interpolant exists for a particular general terminology and a particular set of relevant terms. However, the authors do not address the actual computation of such a uniform interpolant. Up to now, also the bounds on the size of uniform $\mathcal{EL}$ interpolants remain unknown.

In this paper, we propose a worst-case-optimal approach based on proof theory and the theory of formal tree languages to computing a finite uniform $\mathcal{EL}$ interpolant for a general terminology. For this purpose, we introduce regular tree grammars representing subsumees and subsumers of atomic concepts, which, after a sequence of non-terminal replacements, can be transformed into a uniform $\mathcal{EL}$ interpolant of at most triple exponential size, if such a finite uniform $\mathcal{EL}$ interpolant exists for the given terminology and a set of terms. Further, by the means of an example we show that, in the worst-case, no shorter interpolants exist, thereby establishing the triple exponential tight bounds on the size of uniform interpolants in $\mathcal{EL}$.

The paper is structured as follows: In Section 2, we recall the necessary preliminaries on $\mathcal{EL}$ and regular tree languages/grammars. Section 3 formally introduces the notion of inseparability, defines the task of uniform interpolation and provides an example that demonstrates that the smallest uniform interpolants in $\mathcal{EL}$ can be triple exponential in the size of the original knowledge base. In Section 5, we introduce regular tree grammars representing subsumees and subsumers of atomic concepts, which are the basis for computing uniform $\mathcal{EL}$ interpolants as shown in Section 6. In the same section, we also show the upper bound on the size of uniform interpolants. We summarize the contributions in Section 7 and discuss some ideas for future work. Detailed proofs can be found in the appendix of this paper.

2 Preliminaries

Let N_C and N_R be countably infinite and mutually disjoint sets of concept symbols and role symbols. An $\mathcal{EL}$ concept C is defined as

$$C ::= A|\top|C \sqcap C|\exists r.C$$

where A and r range over N_C and N_R, respectively. In the following, we use symbols A, B to denote atomic concepts and C, D to denote arbitrary concepts. A *terminology* or *TBox* consists of *concept inclusion* axioms $C \sqsubseteq D$ and *concept equivalence* axioms $C \equiv D$ used as a shorthand for $C \sqsubseteq D$ and $D \sqsubseteq C$. While knowledge bases in general can also include a specification of individuals with the corresponding concept and role assertions (ABox), in this paper we abstract from ABoxes and concentrate on TBoxes. The signature of an $\mathcal{EL}$ concept C or an axiom α, denoted by $\text{sig}(C)$ or $\text{sig}(\alpha)$, respectively, is the set of concept and role symbols occurring in it. To distinguish between the set of concept symbols and the set of role symbols, we use $\text{sig}_C(C)$ and $\text{sig}_R(C)$, respectively. The signature of a TBox $\mathcal{T}$, in symbols $\text{sig}(\mathcal{T})$ (correspondingly, $\text{sig}_C(\mathcal{T})$ and $\text{sig}_R(\mathcal{T})$), is defined analogously. Next, we recall the semantics of the above introduced DL constructs, which is defined by the means of interpretations. An interpretation $\mathcal{I}$ is given by the domain $\Delta^\mathcal{I}$ and a function $\cdot^\mathcal{I}$ assigning each concept $A \in N_C$ a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$ and each role $r \in N_R$ a subset $r^\mathcal{I}$ of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$. The interpretation of $\top$ is fixed to $\Delta^\mathcal{I}$. The interpretation of an arbitrary $\mathcal{EL}$ concept is defined inductively, i.e., $(C \sqcap D)^\mathcal{I} = C^\mathcal{I} \cap D^\mathcal{I}$ and $(\exists r.C)^\mathcal{I} = \{x \mid (x, y) \in r^\mathcal{I}, y \in C^\mathcal{I}\}$. An interpretation $\mathcal{I}$ satisfies an axiom $C \sqsubseteq D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$. $\mathcal{I}$ is a model of a TBox, if it satisfies all of its axioms. We say that a TBox $\mathcal{T}$ entails an axiom α (in symbols, $\mathcal{T} \models \alpha$), if α is satisfied by all models of $\mathcal{T}$.

Tree Languages and Regular Tree Grammars

A ranked alphabet is a pair ($\mathcal{F}$, Arity) where $\mathcal{F}$ is a finite set and Arity is a mapping from $\mathcal{F}$ into $\mathbb{N}$. $T(\mathcal{F})$ denotes the set of ground terms over the alphabet $\mathcal{F}$. Let $\mathcal{X}_n$ be a set of n variables. A term $C \in T(\mathcal{F}, \mathcal{X}_n)$ containing each variable from $\mathcal{X}_n$ at most once is called a context. We denote by $C(\mathcal{F})$ the set of contexts containing a single variable. A regular tree grammar $G = (S, \mathcal{N}, \mathcal{F}, R)$ is composed of a start symbol S, a set $\mathcal{N}$ of non-terminal symbols (non-terminal symbols have arity 0) with $S \in \mathcal{N}$, a ranked alphabet $\mathcal{F}$ of terminal symbols with a fixed arity such that $\mathcal{F} \cap \mathcal{N} = \emptyset$, and a set R of derivation rules of the form $X \to \beta$ where β is a tree of $T(\mathcal{F} \cup \mathcal{N})$ and $X \in \mathcal{N}$. Given a regular tree grammar $G = (S, \mathcal{N}, \mathcal{F}, R)$, the derivation relation $\to_G$ associated to G is a relation on pairs of terms of $T(\mathcal{F} \cup \mathcal{N})$ such that $s \to_G t$ if and only if there is a rule $X \to \alpha \in R$ and there is a context C such that $s = C[X]$ and $t = C[\alpha]$. The language generated by G, denoted by $L(G)$ is a subset of $T(\mathcal{F})$ which can be reached by successive derivations starting from the start symbol, i.e. $L(G) = \{s \in T \mid S \to^+ s\}$ with $\to^+$ the transitive closure of $\to$. We write $\to$ instead of $\to_G$ when the grammar G is clear from the context. For further details, we refer the reader, for instance, to [5].

3 Uniform Interpolation

Formally, the term uniform interpolation is defined based on the notion of *inseparability*. Two TBoxes, $\mathcal{T}_1$ and $\mathcal{T}_2$, are inseparable w.r.t. a signature Σ if they have the same Σ-consequences, i.e., consequences whose signature is a subset of Σ. Depending on the particular application requirements, the expressivity of those Σ consequences can vary from subsumption queries and instance queries to conjunctive queries. In this paper, we investigate uniform interpolation based on concept inseparability of general $\mathcal{EL}$ terminologies defined analogously to previous work on inseparability, e.g., [8] or [7], as follows:

Definition 1 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two general $\mathcal{EL}$ TBoxes and Σ a signature. $\mathcal{T}_1$ and $\mathcal{T}_2$ are concept-inseparable w.r.t. Σ, in symbols $\mathcal{T}_1 \equiv^c_\Sigma \mathcal{T}_2$, if for all $\mathcal{EL}$ concepts C, D with $sig(C) \cup sig(D) \subseteq \Sigma$ holds $\mathcal{T}_1 \models C \sqsubseteq D$, iff $\mathcal{T}_2 \models C \sqsubseteq D$.*

Given a signature Σ and a TBox $\mathcal{T}$, the aim of uniform interpolation is to determine a TBox $\mathcal{T}'$ with $\text{sig}(\mathcal{T}') \subseteq \Sigma$ such that $\mathcal{T} \equiv^c_\Sigma \mathcal{T}'$. $\mathcal{T}'$ is also called a *uniform $\mathcal{EL}$ Σ-interpolant* of $\mathcal{T}$. In practise, uniform interpolants are required to be finite, i.e., expressible by a finite set of finite axioms using only the language constructs of $\mathcal{EL}$. As demonstrated by the following example, in the presence of cyclic concept inclusions, a finite uniform $\mathcal{EL}$ Σ-interpolant might not exist for a particular TBox $\mathcal{T}$ and a particular Σ.

Example 1 *Consider uniform interpolants of the TBox $\mathcal{T} = \{A' \sqsubseteq A, A \sqsubseteq A'', A \sqsubseteq \exists r.A, \exists s.A \sqsubseteq A\}$. w.r.t. $\Sigma = \{s, r, A', A''\}$. We obtain an infinite chain of consequences $A' \sqsubseteq \exists r.\exists r.\exists r....A''$ and $\exists s.\exists s.\exists s....A' \sqsubseteq A''$ containing nested existential quantifiers of unbounded depth.*

It is interesting that, while deciding the existence of uniform interpolants in $\mathcal{EL}$ [9] is one exponential less complex than the same decision problem for the more complex logic $\mathcal{ALC}$ [11], the size of uniform interpolants remains triple-exponential due to the unavailability of disjunction. We demonstrate that this is in fact the lower bound by the means of the following example (obtained by a slight modification of the corresponding example given in [10] originally demonstrating a double exponential lower bound in the context of conservative extensions).

Example 2 *The $\mathcal{EL}$ TBox $\mathcal{T}_n$ for a natural number n is given by*

$$A_1 \sqsubseteq \overline{X_0} \sqcap ... \sqcap \overline{X_{n-1}} \quad (1)$$

$$A_2 \sqsubseteq \overline{X_0} \sqcap ... \sqcap \overline{X_{n-1}} \quad (2)$$

$$\sqcap_{\sigma\in\{r,s\}} \exists\sigma.(\overline{X_i} \sqcap X_0 \sqcap ... \sqcap X_{i-1}) \sqsubseteq X_i \qquad i < n \quad (3)$$

$$\sqcap_{\sigma\in\{r,s\}} \exists\sigma.(X_i \sqcap X_0 \sqcap ... \sqcap X_{i-1}) \sqsubseteq \overline{X_i} \qquad i < n \quad (4)$$

$$\sqcap_{\sigma\in\{r,s\}} \exists\sigma.(\overline{X_i} \sqcap \overline{X_j}) \sqsubseteq \overline{X_i} \quad j < i < n \quad (5)$$

$$\sqcap_{\sigma\in\{r,s\}} \exists\sigma.(X_i \sqcap \overline{X_j}) \sqsubseteq X_i \quad j < i < n \quad (6)$$

$$X_0 \sqcap ... \sqcap X_{n-1} \sqsubseteq B \quad (7)$$

In the above TBox, Axiom 3 insures that an unset bit will be set in the successor number, if all bits before it are already set. The subsequent axiom 4 insures that a set bit will be unset in the successor number, if all bits before it are also set. Axioms 5 and 6 insure that in all other cases, bits do not switch. For instance, Axioms 5 states that, if any bit before bit i is unset yet, then bit i will remain unset also in the successor number.

If we now consider sets $\mathcal{C}_i$ of concept descriptions inductively defined by $\mathcal{C}_0 = \{A_1, A_2\}$,$\mathcal{C}_{i+1} = \{\exists r.C_1 \sqcap \exists s.C_2 \mid C_1, C_2 \in \mathcal{C}_i\}$, then we find that $|\mathcal{C}_{i+1}| = |\mathcal{C}_i|^2$ and consequently $|\mathcal{C}_i| = 2^{(2^i)}$. Thus, the set $\mathcal{C}_{2^n-1}$ contains triply exponentially many different concepts, each of which is doubly exponential in the size of $\mathcal{T}_n$ (intuitively, we obtain concepts having the shape of binary trees of exponential depth, thus having doubly exponentially many leaves, each of which can be endowed with A_1 or A_2, which gives rise to triply exponentially many different such trees). Then it can be shown *that for each concept $C \in \mathcal{C}_{2^n-1}$ holds $\mathcal{T}_n \models C \sqsubseteq B$ and that there cannot be a smaller uniform interpolant w.r.t. the signature $\Sigma = \{A_1, A_2, B, r, s\}$ than the one containing all these GCIs (for a proof, see Appendix B).*

Hence we have found a class $\mathcal{T}_n$ of TBoxes giving rise to uniform interpolants of triple-exponential size in terms of the original TBox. In the following, we show that this is also an upper bound by providing a procedure for computing uniform interpolants with a triple-exponentially bounded output.

4 Normalization

Similarly to other proof-theoretic approaches [1, 6, 7], we will make use of normalizations that restrict the syntactic form of TBoxes. We decompose complex axioms into syntactically simpler ones. The decomposition is realized recursively by replacing sub-expressions $C_1 \sqcap ... \sqcap C_n$ and $\exists r.C$ by fresh concept symbols until each axiom in the TBox $\mathcal{T}$ is one of $\{A \sqsubseteq B, A \equiv B_1 \sqcap ... \sqcap B_n, A \equiv \exists r.B\}$, where $A, B, B_i \in \text{sig}_C(\mathcal{T}) \cup \{\top\}$ and $r \in \text{sig}_R(\mathcal{T})$. For this purpose, we introduce a minimal required set of fresh concept symbols N_D and the corresponding definition axioms $\{A' \equiv C' \mid A' \in N_D\}$ for each $A' \in N_D$ and the corresponding concept C' replaced by A'.

In what follows, we assume that knowledge bases are normalized and refer to $\text{sig}_C(\mathcal{T}) \cup N_D$ as $\text{sig}_C(\mathcal{T})$. Since concept symbols in N_D are fresh, they do not appear in Σ. W.l.o.g., in what follows we assume that $\mathcal{EL}$ concepts do not contain any equivalent concepts in conjunctions and that equivalent concept symbols have been replaced by a single representative of the corresponding equivalence class. The following lemma postulates the close semantic relation between a TBox and its normalization.

Lemma 1 *Any $\mathcal{EL}$ TBox $\mathcal{T}$ can be extended into a normalized TBox $\mathcal{T}'$ such that each model of $\mathcal{T}'$ is a model of $\mathcal{T}$ and each model of $\mathcal{T}$ can be extended into a model of $\mathcal{T}'$.*

Proof Sketch. All concepts in N_D are defined, i.e., their meaning is uniquely determined by the meaning of subconcepts (concepts that occur in $\mathcal{T}$) of the original TBox $\mathcal{T}$. □

The following lemma motivates the usefulness of the normalization for the computation of uniform interpolants. In particular, it allows us to restrict the information necessary for the uniform interpolation to the sets of subsumers and subsumees of all atomic concepts in the TBox.

Lemma 2 *Let $\mathcal{T}$ be normalized $\mathcal{EL}$ TBox and C, D two $\mathcal{EL}$ concepts with $sig(C) \cup sig(D) \subseteq sig(\mathcal{T})$ such that $\mathcal{T} \models C \sqsubseteq D$. For any $A \in sig_C(\mathcal{T})$, let $\texttt{Pre}(A) = \{M \subseteq sig_C(\mathcal{T}) \mid \mathcal{T} \models \bigsqcap_{B_i \in M} B_i \sqsubseteq A\}$. W.l.o.g., assume that*

$$C = \bigsqcap_{1 \le j \le n} A_j \sqcap \bigsqcap_{1 \le k \le m} \exists r_k.E_k$$

for $A_j \in sig_C(\mathcal{T})$ and $r_k \in sig_R(\mathcal{T})$, E_k $\mathcal{EL}$ concepts with $sig(E_k) \subseteq sig(\mathcal{T})$ for $1 \le k \le m$. For all conjuncts D_i of D, the following is true: If $D_i \in sig_C(\mathcal{T})$, there is a set $M \in \texttt{Pre}(D_i)$ of atomic concepts such that for each element B of M holds at least one of the conditions [A1]-[A2]:

(A1) There is an A_j in C such that $A_j = B$.

(A2) There are r_k, E_k and there exists $B' \in sig_C(\mathcal{T})$ such that $\mathcal{T} \models E_k \sqsubseteq B'$ and $B \equiv \exists r_k.B' \in \mathcal{T}$.

If $D_i = \exists r'.D'$ for $r' \in sig_R(\mathcal{T})$ and D' an $\mathcal{EL}$ concept , at least one of the conditions [A3]-[A4] holds:

(A3) There are r_k, E_k such that $r_k = r'$ and $\mathcal{T} \models E_k \sqsubseteq D'$.

(A4) There is a $B \in sig_C(\mathcal{T})$ such that $\mathcal{T} \models B \sqsubseteq \exists r'.D'$ and $\mathcal{T} \models C \sqsubseteq B$.

Proof. The proof is based on a Gentzen-style calculus for $\mathcal{EL}$ complete for subsumptions between arbitrary $\mathcal{EL}$ concepts shown in Fig. 1. We consider all rules, that could have been the last rule applied in order to derive the above sequent and show the lemma by induction on the length of the proof. □

$$\frac{}{C \sqsubseteq C}(\textsc{Ax}) \quad \frac{}{C \sqsubseteq \top}(\textsc{AxTop})$$

$$\frac{D \sqsubseteq E}{C \sqcap D \sqsubseteq E}(\textsc{AndL})$$

$$\frac{C \sqsubseteq E \quad C \sqsubseteq D}{C \sqsubseteq D \sqcap E}(\textsc{AndR})$$

$$\frac{C \sqsubseteq D}{\exists r.C \sqsubseteq \exists r.D}(\textsc{Ex})$$

$$\frac{C \sqsubseteq E \quad E \sqsubseteq D}{C \sqsubseteq D}(\textsc{Cut})$$

Figure 1. Gentzen-style proof system for general $\mathcal{EL}$ terminologies.

Lemma 2 allows us, on the one hand, to prove the completeness of grammars introduced in the next section, and, on the other hand, to show that the TBox computed in Section 6 by combining subsumees and subsumers into subsumption axioms indeed entails all Σ-consequences of $\mathcal{T}$.

5 Grammar Representation of Subsumees and Subsumers

In order to obtain a finite uniform interpolant from the infinite sets of subsumees and subsumers, a finite representation for these sets is required. In this section, we show how, for a signature Σ, the sets of Σ-subsumees and Σ-subsumers of each atomic concept in a normalized $\mathcal{EL}$ TBox $\mathcal{T}$ can be described as languages generated by regular tree grammars on ranked unordered trees with finite sets of derivation rules later on transformed into a finite uniform interpolant. For the definition of the grammars, we uniquely represent each atomic concept $A \in \text{sig}_C(\mathcal{T})$ by a non-terminal $\mathtt{n}_A$ (and denote the set of all non-terminals by $\mathcal{N}^{\mathcal{T}} = \{\mathtt{n}_x | x \in \text{sig}_C(\mathcal{T}) \cup \{\top\}\}$). In what follows, we use the ranked alphabet $\mathcal{F} = (\text{sig}_C(\mathcal{T}) \cap \Sigma) \cup \{\top\} \cup \{\exists r \mid r \in \text{sig}_R(\mathcal{T}) \cap \Sigma\} \cup \{\sqcap_i \mid i \le n\}$, where atomic concepts in $\text{sig}_C(\mathcal{T}) \cap \Sigma$ are constants, $\exists r$ for $r \in \text{sig}_R(\mathcal{T}) \cap \Sigma$ are unary functions and $\sqcap_i$ are functions of the arity i bounded by $n = |\text{sig}_C(\mathcal{T})| \cdot (|\text{sig}_R(\mathcal{T})| + 1)$, i.e., the number of all possible simple concepts in $\mathcal{T}$ (atomic concepts and all existential restrictions on atomic concepts). The restriction to the maximum arity of n is w.l.o.g., since we can always split longer conjunctions into a nested conjunction with at most n elements in each sub-expression. In the following, it will be convenient to simply write $\sqcap$ if the arity of the corresponding function is clear from the context. Clearly, every $\mathcal{EL}$ concept C with $\text{sig}(C) \subseteq \Sigma$ and at most n conjuncts in each sub-expression has a unique representation by the means of the above functions. We denote such a term representation of C using $\mathcal{F}$ by t_C.

In what follows, we use a substituting function $\sigma_{\mathcal{T},\mathcal{F}} : \{C \mid \mathrm{sig}(C) \subseteq \mathrm{sig}(\mathcal{T})\} \rightarrow T(\mathcal{F}, \mathcal{N}^{\mathcal{T}})$ by $\sigma_{\mathcal{T},\mathcal{F}}(C) = t_C\{\mathfrak{n}_\top/\top, \mathfrak{n}_{B_1}/B_1, ..., \mathfrak{n}_{B_n}/B_n\}$, where $B_1, ..., B_n$ are all atomic sub-expressions of C. Note that $\sigma_{\mathcal{T},\mathcal{F}}$ is injective, therefore, its inverse is also a function. If the TBox and the set of non-terminals are clear from the context, we will denote such a representation of a concept C simply by $\sigma(C)$, and its inverse by $\sigma^-(t)$ for $t \in T(\mathcal{F}, \mathcal{N}^{\mathcal{T}})$. In the following we will assume $\sigma^-(t)$ to be extended to partially ground terms and ground terms.

Since concepts are represented as terms, we extend the generated languages by associative variants of concept expressions. For this purpose, in addition to the TBox axioms and classification results, we include in our grammars the subsumees and subsumers of each atomic concept having the form of simple conjunctions, i.e., conjunctions of simple concepts. As we will see in the next section, to obtain a uniform interpolant and derive the corresponding upper bound, in the case of subsumees, it is sufficient to capture all associative variants of subsumees not being obtained by adding arbitrary conjuncts to arbitrary sub-expressions (rule ANDL in Fig. 1). In fact, in general, adding arbitrary conjuncts to arbitrary sub-expressions allows us to obtain subsumees being conjunctions of unbounded size, which would cause the corresponding language to contain terms with $\sqcap$-functions of unbounded arity and make the definition of the grammar unnecessary complex. Therefore, we do not include such subsumees into our grammars. For this reason, it is sufficient in the case of subsumees to consider conjunctions of atomic concepts only, denoted by $\texttt{Pre}(A) = \{M \subseteq \mathrm{sig}_C(\mathcal{T}) \mid \mathcal{T} \models \bigsqcap_{B_i \in M} B_i \sqsubseteq A\}$.

In contrast to that, to be able to derive the upper bound, we have to include all subsumers into our grammars. Since weakening of subsumers (see rule ANDR in Fig. 1) does not require $\sqcap$-functions of unbounded arity, this can be done by the means of a minor extension: in addition to conjunctions of atomic concepts, we take into account existential restrictions with atomic concepts, formed from the elements of the set $\texttt{Post}_{\text{Base}}(A) = \{A' \in \mathrm{sig}_C(\mathcal{T}) \cup \{\top\} \mid \mathcal{T} \models A \sqsubseteq A'\} \cup \{\exists r.A' \mid A' \in \mathrm{sig}_C(\mathcal{T}) \cup \{\top\}, \mathcal{T} \models A \sqsubseteq \exists r.A', r \in \Sigma\}$ and $\texttt{Post}(A) = 2^{\texttt{Post}_{\text{Base}}(A)}$. Thereby, we obtain the following definition.

Definition 2 *Let $\mathcal{T}$ be a normalized $\mathcal{EL}$ TBox, Σ a signature. Further, let $\texttt{Pre}(A) = \{M \subseteq sig_C(\mathcal{T}) \mid \mathcal{T} \models \bigsqcap_{B_i \in M} B_i \sqsubseteq A\}$, $\texttt{Post}_{Base}(A) = \{A' \in sig_C(\mathcal{T}) \cup \{\top\} \mid \mathcal{T} \models A \sqsubseteq A'\} \cup \{\exists r.A' \mid A' \in sig_C(\mathcal{T}) \cup \{\top\}, \mathcal{T} \models A \sqsubseteq \exists r.A', r \in \Sigma\}$ and $\texttt{Post}(A) = 2^{\texttt{Post}_{Base}(A)}$. Further, for each $B \in sig_C(\mathcal{T})$, let $R^{\sqsupseteq}$ be given by*

- **(GL1)** $\mathfrak{n}_B \rightarrow B$ *if* $B \in \Sigma$,
- **(GL2)** $\mathfrak{n}_B \rightarrow \mathfrak{n}_{B'}$ *for all* $\{B'\} \in \texttt{Pre}(B)$,
- **(GL3)** $\mathfrak{n}_B \rightarrow \sqcap(\mathfrak{n}_{B'_1}, ..., \mathfrak{n}_{B'_n})$ *for all* $\{B'_1, ..., B'_n\} \in \texttt{Pre}(B)$ *with* $n \geq 1$,
- **(GL4)** $\mathfrak{n}_B \rightarrow \exists r(\mathfrak{n}_{B'})$ *for all* B' *with* $B \equiv \exists r.B' \in \mathcal{T}$ *and* $r \in sig_R(\mathcal{T}) \cap \Sigma$.

Let $R^{\sqsubseteq}$ be given for all $B \in sig_C(\mathcal{T}) \cup \{\top\}$ by

- **(GR1)** $\mathfrak{n}_B \rightarrow B$ *if* $B \in \Sigma \cup \{\top\}$,
- **(GR2)** $\mathfrak{n}_B \rightarrow \sigma(C)$ *for all* $\{C\} \in \texttt{Post}(B)$,
- **(GL3)** $\mathfrak{n}_B \rightarrow \sqcap(\sigma(C'_1), ..., \sigma(C'_n))$ *for all* $\{C'_1, ..., C'_n\} \in \texttt{Post}(B)$ *with* $n \geq 1$.

For each $A \in sig_C(\mathcal{T})$, the regular tree grammar $G^{\sqsupseteq}(\mathcal{T}, \Sigma, A)$ is then given by $(\mathfrak{n}_A, \mathcal{N}^{\mathcal{T}}, \mathcal{F}, R^{\sqsupseteq})$, and the regular tree grammar $G^{\sqsubseteq}(\mathcal{T}, \Sigma, A)$ is given by $(\mathfrak{n}_A, \mathcal{N}^{\mathcal{T}}, \mathcal{F}, R^{\sqsubseteq})$.

We denote the set of tree grammars $\{G^{\sqsupseteq}(\mathcal{T}, \Sigma, A) \mid A \in \mathrm{sig}_C(\mathcal{T})\}$ by $\mathbb{G}^{\sqsupseteq}(\mathcal{T}, \Sigma)$ and the set $\{G^{\sqsubseteq}(\mathcal{T}, \Sigma, A) \mid A \in \mathrm{sig}_C(\mathcal{T})\}$ by $\mathbb{G}^{\sqsubseteq}(\mathcal{T}, \Sigma)$.

Since $\mathrm{sig}(\mathcal{T})$ is finite, all elements of $\texttt{Pre}$ and $\texttt{Post}$ can be effectively computed. For the construction of grammars the following result holds.

Theorem 1 *Let $\mathcal{T}$ be a normalized $\mathcal{EL}$ TBox, Σ a signature. $\mathbb{G}^{\sqsupseteq}(\mathcal{T}, \Sigma)$ and $\mathbb{G}^{\sqsubseteq}(\mathcal{T}, \Sigma)$ can be computed from $\mathcal{T}$ in exponential time and are exponentially bounded in the size of $\mathcal{T}$.*

Proof Sketch. The exponentially bounded size and time hold basically due to the exponential number of elements in $\texttt{Pre}$ and $\texttt{Post}$ and tractable reasoning in $\mathcal{EL}$[1]. □

The following example demonstrates the grammar construction.

Example 3 *For $\mathcal{T}$ and Σ from Example 1, we obtain a normalized TBox $\mathcal{T}' = \{A' \sqsubseteq A, A \sqsubseteq A'', A \sqsubseteq B, B \equiv \exists r.A, B' \equiv \exists s.A, B' \sqsubseteq A\}$, which yields $\texttt{Pre} = \{(A, \{A', B'\}), (A'', \{A', B', A\}), (A', \{\}), (B, \{A', A\}), (B', \{\})\}$, $\texttt{Post}_{Base} = \{(A, \{A'', B, \top, \exists r(\mathfrak{n}_A), \exists r(\mathfrak{n}_\top)\}), (A', \{A, A'', B, \top, \exists r(\mathfrak{n}_A), \exists r(\mathfrak{n}_\top)\}), (B, \{\top, \exists r(\mathfrak{n}_A), \exists r(\mathfrak{n}_\top)\}), (A'', \{\top\}), (B', \{A'', A, \top, \exists s(\mathfrak{n}_A), \exists s(\mathfrak{n}_\top)\})\}$ and the following set of transitions for $R^{\sqsupseteq}$:*

$$\mathfrak{n}_B \rightarrow \mathfrak{n}_A$$

$$\begin{array}{rl}
\mathfrak{n}_{A''} \rightarrow \mathfrak{n}_{A'} & \mathfrak{n}_A \rightarrow \mathfrak{n}_{B'} \\
\mathfrak{n}_{A''} \rightarrow \mathfrak{n}_A & \mathfrak{n}_A \rightarrow \mathfrak{n}_{A'} \\
\mathfrak{n}_{A''} \rightarrow \mathfrak{n}_{B'} & \mathfrak{n}_B \rightarrow \mathfrak{n}_{A'} \\
\mathfrak{n}_{A''} \rightarrow A'' & \mathfrak{n}_{A'} \rightarrow A' \\
\mathfrak{n}_{B'} \rightarrow \exists s(\mathfrak{n}_A) & \mathfrak{n}_B \rightarrow \exists r(\mathfrak{n}_A) \\
\mathfrak{n}_A \rightarrow \sqcap(\mathfrak{n}_{A'}, \mathfrak{n}_{B'}) & \mathfrak{n}_B \rightarrow \sqcap(\mathfrak{n}_{A'}, \mathfrak{n}_A) \\
\mathfrak{n}_{A''} \rightarrow \sqcap(\mathfrak{n}_{A'}, \mathfrak{n}_A) & \mathfrak{n}_{A''} \rightarrow \sqcap(\mathfrak{n}_A, \mathfrak{n}_{B'}) \\
\mathfrak{n}_{A''} \rightarrow \sqcap(\mathfrak{n}_{A'}, \mathfrak{n}_{B'}) & \mathfrak{n}_{A''} \rightarrow \sqcap(\mathfrak{n}_A, \mathfrak{n}_{A'}, \mathfrak{n}_{B'})
\end{array}$$

For $R^{\sqsubseteq}$, we obtain $\mathfrak{n} \rightarrow \mathfrak{n}_\top$ for all $\mathfrak{n} \in \mathcal{N}$ and

$$\begin{array}{rl}
\mathfrak{n}_{A''} \rightarrow A'' & \mathfrak{n}_\top \rightarrow \top \\
\mathfrak{n}_{A'} \rightarrow A' & \mathfrak{n}_{A'} \rightarrow \mathfrak{n}_B \\
\mathfrak{n}_A \rightarrow \mathfrak{n}_{A''} & \mathfrak{n}_{A'} \rightarrow \mathfrak{n}_A \\
\mathfrak{n}_A \rightarrow \mathfrak{n}_B & \mathfrak{n}_{A'} \rightarrow \mathfrak{n}_{A''} \\
\mathfrak{n}_{B'} \rightarrow \mathfrak{n}_A & \mathfrak{n}_{B'} \rightarrow \mathfrak{n}_{A''} \\
\mathfrak{n}_{B'} \rightarrow \exists s(\mathfrak{n}_A) & \mathfrak{n}_B \rightarrow \exists r(\mathfrak{n}_A) \\
\mathfrak{n}_A \rightarrow \exists r(\mathfrak{n}_A) & \mathfrak{n}_{A'} \rightarrow \exists r(\mathfrak{n}_A) \\
\mathfrak{n}_{B'} \rightarrow \exists s(\mathfrak{n}_\top) & \mathfrak{n}_B \rightarrow \exists r(\mathfrak{n}_\top) \\
\mathfrak{n}_A \rightarrow \exists r(\mathfrak{n}_\top) & \mathfrak{n}_{A'} \rightarrow \exists r(\mathfrak{n}_\top)
\end{array}$$

Additionally, $R^{\sqsubseteq}$ contains rules for conjunctions of all elements of $\texttt{Post}_{Base}$ corresponding to $(\mathbf{GR3})$*, which we do not give for space reasons.*

By applying the rules $\mathfrak{n}_A \rightarrow \mathfrak{n}_{B'}, \mathfrak{n}_{B'} \rightarrow \exists s(\mathfrak{n}_A)$ contained in $R^{\sqsupseteq}$ n times, we obtain a term $\exists s(\exists s(... \exists s(A)))$ of depth n, which represents the corresponding subsumee of A of the same depth.

5.1 Grammar Properties

The following theorem states that the grammars derive only terms representing Σ-subsumees and Σ-subsumers of the corresponding atomic concept.

Theorem 2 *Let $\mathcal{T}$ be a normalized $\mathcal{EL}$ TBox, Σ a signature and $A \in sig_C(\mathcal{T})$.*

1. *For each $t \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$, there is a concept C with $t_C = t$ and $sig(C) \subseteq \Sigma$ such that $\mathcal{T} \models C \sqsubseteq A$.*

2. *For each $t \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$, there is a concept C with $t_C = t$ and $sig(C) \subseteq \Sigma$ such that $\mathcal{T} \models A \sqsubseteq C$.*

Proof Sketch. The theorem is proved by an easy induction on the maximal nesting depth of functions in t using the rules given in Definition 2. ◻

As discussed above, for the completeness of the grammar generating subsumees, we only guarantee to capture all associative variants of concepts not being obtained by adding arbitrary conjuncts to arbitrary sub-expressions.

Theorem 3 *Let $\mathcal{T}$ be a normalized $\mathcal{EL}$ TBox, Σ a signature and $A \in sig_C(\mathcal{T})$.*

1. *For each C with $sig(C) \subseteq \Sigma$ such that $\mathcal{T} \models C \sqsubseteq A$ there is a concept C' such that C can be obtained from C' by adding arbitrary conjuncts to arbitrary sub-expressions and $t_{C'} \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$.*
2. *For each D with $sig(D) \subseteq \Sigma$ such that $\mathcal{T} \models A \sqsubseteq D$ holds: $t_D \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$.*

Proof Sketch. The theorem is proved by induction on the role depth of C using the properties of the normalization, for instance, stated in Lemmas 2, in addition to Definition 2. ◻

6 From Grammars to Uniform Interpolants

For the construction of a uniform interpolant, we make use of the results stated in Lemma 2, which, in combination with the introduced normalization imply that, knowing the subsumees and subsumers of atomic concepts in normalized terminologies is sufficient to derive all subsumptions between any complex concepts. In order to obtain a corresponding TBox from a pair of grammars, for all $\mathfrak{n}_B$ occurring on the right-hand sides of the transition rules must hold: $B \in \Sigma \cup \{\top\}$. If the latter is the case, we can apply the inverse substitution $\sigma^-(t)$ to obtain axioms defining subsumers and subsumees of atomic concepts. Otherwise, we first need to eliminate all non-terminals not from $\mathcal{N}^\Sigma = \{\mathfrak{n}_B \mid B \in \Sigma \cup \{\top\}\}$ within the right-hand sides of the corresponding rules. In principle, we can substitute any such non-terminal $\mathfrak{n} \notin \mathcal{N}^\Sigma$ by the right-hand sides of the corresponding rules for $\mathfrak{n}$ without any change to the generated language. However, in the general case, such a sequence of substitutions does not have to be finite. In the following, we investigate the bounds for the number of such substitution steps required to obtain a uniform interpolant.

For a concept C, let $d(C)$ denote the maximal role depth within C. For a TBox $\mathcal{T}$, $d(\mathcal{T}) = \mathtt{max}\{d(C) \mid C$ is a sub-expression of $\mathcal{T}\}$. The following lemma postulates a bound on the role depth of minimal uniform $\mathcal{EL}$ interpolants:

Lemma 3 *Let $\mathcal{T}$ be a normalized $\mathcal{EL}$ TBox, Σ a signature. Let $def(\mathcal{T})$ be the number of definitions in $\mathcal{T}$. The following statements are equivalent:*

1. *There exists a uniform $\mathcal{EL}$ Σ-interpolant of $\mathcal{T}$.*
2. *There exists a uniform $\mathcal{EL}$ Σ-interpolant $\mathcal{T}'$ of $\mathcal{T}$ and $d(\mathcal{T}') \leq 2^{4\cdot(|sig_C(\mathcal{T})|+def(\mathcal{T}))} + 1$.*

Proof Sketch. In a normalized TBox $\mathcal{T}$, the number of sub-expressions[1] is $|\text{sig}_C(\mathcal{T})| + \text{def}(\mathcal{T})$. Therefore, we can replace the last statement of Condition 2 by $d(\mathcal{T}') \leq 2^{2\cdot n} + 1$, where n is twice the number of sub-expressions within $\mathcal{T}$. Then, the lemma follows from Conditions (1) and (4) of Lemma 55 in [9]. ◻

We can eliminate all non-terminals not from $\mathcal{N}^\Sigma$ within the given role depth by replacing them in each rule by the corresponding right-hand sides, thereby obtaining a set of grammars that can be transformed into a uniform $\mathcal{EL}$ Σ-interpolant using the inverse substitution $\sigma^-(t)$.

Definition 3 *For a normalized $\mathcal{EL}$ TBox $\mathcal{T}$ and a signature Σ, let*

- *$R_0^{\sqsupseteq} = R^{\sqsupseteq}$ and $R_0^{\sqsubseteq} = R^{\sqsubseteq}$.*
- *$R_{i+1}^{\bowtie} = \{\mathfrak{n} \to t(t_1', ..., t_n') \mid \mathfrak{n} \to t(\mathfrak{n}_1, ..., \mathfrak{n}_n) \in R_i^{\bowtie}, 1 \leq j \leq n, t_j' = \mathfrak{n}_j$ if $\mathfrak{n}_j \in \mathcal{N}^\Sigma$ and $t_j' \in \{t' \mid \mathfrak{n}_j \to t' \in R_0^{\bowtie}\}$ for $\mathfrak{n}_j \notin \mathcal{N}^\Sigma\}$ with $\bowtie \in \{\sqsupseteq, \sqsubseteq\}$.*

For an $A \in sig_C(\mathcal{T})$, let $G_i^{\sqsupseteq} = (\mathfrak{n}_A, \mathcal{N}^\mathcal{T}, \mathcal{F}, R_i^{\sqsupseteq})$ and $G_i^{\sqsubseteq} = (\mathfrak{n}_A, \mathcal{N}^\mathcal{T}, \mathcal{F}, R_i^{\sqsubseteq})$. $\mathbb{G}_i^{\sqsupseteq}(\mathcal{T}, \Sigma)$ is then given by $\{G_i^{\sqsupseteq}(\mathcal{T}, \Sigma, A) \mid A \in sig_C(\mathcal{T})\}$ and $\mathbb{G}_i^{\sqsubseteq}(\mathcal{T}, \Sigma)$ by $\{G_i^{\sqsubseteq}(\mathcal{T}, \Sigma, A) \mid A \in sig_C(\mathcal{T})\}$.

Let $N = 2^{4\cdot(|\text{sig}_C(\mathcal{T})|+\text{def}(\mathcal{T}))} + 1$. Given a pair of grammar sets $\mathbb{G}_N^{\sqsupseteq}(\mathcal{T}, \Sigma), \mathbb{G}_N^{\sqsubseteq}(\mathcal{T}, \Sigma)$ for a TBox $\mathcal{T}$ and a signature Σ, we can compute a uniform $\mathcal{EL}$ Σ-interpolant of $\mathcal{T}$ as follows.

Definition 4 *Let $\mathcal{T}$ be a normalized $\mathcal{EL}$ TBox, Σ a signature and $N = 2^{4\cdot(|sig_C(\mathcal{T})|+def(\mathcal{T}))} + 1$. Further, let $\mathbb{G}_1 = \mathbb{G}_N^{\sqsupseteq}(\mathcal{T}, \Sigma), \mathbb{G}_2 = \mathbb{G}_N^{\sqsubseteq}(\mathcal{T}, \Sigma)$ with $R_1 = R_N^{\sqsupseteq}$ and $R_2 = R_N^{\sqsubseteq}$. Then, $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) =$*

$$\{\sigma^-(t) \sqsubseteq A \mid A \in \Sigma, \mathfrak{n}_A \to t \in R_1, t \in T(\mathcal{F}, \mathcal{N}^\Sigma)\} \cup$$
$$\{A \sqsubseteq \sigma^-(t) \mid A \in \Sigma, \mathfrak{n}_A \to t \in R_2, t \in T(\mathcal{F}, \mathcal{N}^\Sigma)\} \cup$$
$$\{\sigma^-(t_1) \sqsubseteq \sigma^-(t_2) \mid \mathfrak{n} \notin \mathcal{N}_\Sigma, \mathfrak{n} \to t_1 \in R_1, \mathfrak{n} \to t_2 \in R_2, t_1, t_2 \in T(\mathcal{F}, \mathcal{N}^\Sigma)\}.$$

Clearly, the construction terminates, if $\mathbb{G}_1$ and $\mathbb{G}_2$ are finite. The size of the resulting TBox $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma)$ is bounded polynomially by the size of $\mathbb{G}_1, \mathbb{G}_2$. Moreover, $\text{sig}(\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma)) \subseteq \Sigma$, since each $t, t_1, t_2 \in T(\mathcal{F}, \mathcal{N}^\Sigma)$, $\sigma^-(t) \subseteq \text{sig}(\mathcal{T})$ and $\mathcal{F} \cap (\text{sig}(\mathcal{T}) \setminus \Sigma) = \emptyset$. We obtain the following result concerning the size of uniform $\mathcal{EL}$ Σ-interpolants of $\mathcal{T}$.

Theorem 4 *Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox and Σ a signature. The following statements are equivalent:*

1. *There exists a uniform $\mathcal{EL}$ Σ-interpolant of $\mathcal{T}$.*
2. $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \equiv_\Sigma^c \mathcal{T}$
3. *There exists a uniform $\mathcal{EL}$ Σ-interpolant $\mathcal{T}'$ with $|\mathcal{T}'| \in O(2^{2^{2^{|\mathcal{T}|}}})$.*

Proof. The non-trivial parts of the proof are implications $1 \Rightarrow 2$ and $2 \Rightarrow 3$.

$1 \Rightarrow 2$: By Definition 1, the statement $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \equiv_\Sigma^c \mathcal{T}$ consists of two directions: (1) for all $\mathcal{EL}$ concepts C, D with $\text{sig}(C) \cup \text{sig}(D) \subseteq \Sigma$ holds $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \models C \sqsubseteq D \Rightarrow \mathcal{T} \models C \sqsubseteq D$ and (2) for all $\mathcal{EL}$ concepts C, D with $\text{sig}(C) \cup \text{sig}(D) \subseteq \Sigma$ holds $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \models C \sqsubseteq D \Leftarrow \mathcal{T} \models C \sqsubseteq D$.

(1) The first direction follows from Theorem 2 and Definition 4, which does not introduce any consequences not being consequences of $\mathcal{T}$.

(2) For the second direction, assume that there exists a uniform $\mathcal{EL}$ Σ-interpolant of $\mathcal{T}$. Then, by Lemma 3, there exists a uniform $\mathcal{EL}$ Σ-interpolant $\mathcal{T}'$ of $\mathcal{T}$ with $d(\mathcal{T}') \leq N$.

[1] In a conjunction, only the concepts not being a conjunction itself are considered as proper sub-expressions. Therefore, a conjunction with n elements has n proper sub-expressions.

It is sufficient to show that for each $C \sqsubseteq D \in \mathcal{T}'$ holds $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \models C \sqsubseteq D$. Assume that $C \sqsubseteq D \in \mathcal{T}'$. Then, $\mathcal{T} \models C \sqsubseteq D$ and we prove by induction on maximal role depth of C, D that also $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \models C \sqsubseteq D$. W.l.o.g., let $D = \bigsqcap_{1 \leq i \leq l} D_i$ and

$$C = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.E_k$$

with $A_j \in \Sigma \cap \mathrm{sig}_C(\mathcal{T})$ for $1 \leq j \leq n$, $r_k \in \Sigma \cap \mathrm{sig}_R(\mathcal{T})$ for $1 \leq k \leq m$ and E_k with $1 \leq k \leq m$ a set of $\mathcal{EL}$ concepts such that $\mathrm{sig}(E_k) \subseteq \Sigma$. Clearly, $\mathcal{T} \models C \sqsubseteq D$, iff $\mathcal{T} \models C \sqsubseteq D_i$ for all i with $1 \leq i \leq l$.

- If $D_i = A \in \Sigma$, then, it follows from Theorem 3 that there is a concept C' such that C can be obtained from C' by adding arbitrary conjuncts to arbitrary sub-expressions with $t_{C'} \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$. Since $d(C) \leq N$ and C has been obtained from C' by weakening, also $d(C') \leq N$. Therefore, $t_{C'} \in L(G^{\sqsupseteq}_N(\mathcal{T}, \Sigma, A))$, and $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \models C \sqsubseteq D_i$.
- If $D_i = \exists r.D'$ for some r, D', then, by Lemma 2, one of the following is true:

(A3) There are r_k, E_k in C such that $r_k = r$ and $\mathcal{T} \models E_k \sqsubseteq D'$. Since $d(E_k) < N$ and $d(D') < N$, by induction hypothesis holds $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \models E_k \sqsubseteq D'$. It follows that $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \models \exists r_k.E_k \sqsubseteq D_i$ and $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \models C \sqsubseteq D_i$.

(A4) There is $B \in \mathrm{sig}_C(\mathcal{T})$ of $\mathcal{T}$ such that $\mathcal{T} \models B \sqsubseteq \exists r.D'$ and $\mathcal{T} \models C \sqsubseteq B$. Then,

- it follows from Theorem 3 that there is a concept C' such that C can be obtained from C' by adding arbitrary conjuncts to arbitrary sub-expressions with $t_{C'} \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, B))$. Since $d(C) \leq N$ and C has been obtained from C' by weakening, also $d(C') \leq N$. Therefore, $t_{C'} \in L(G^{\sqsupseteq}_N(\mathcal{T}, \Sigma, B))$
- it follows from Theorem 3 that $t_{\exists r.D'} \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, B))$. Since $d(\exists r.D') \leq N$, it follows that $t_{\exists r.D'} \in L(G^{\sqsubseteq}_N(\mathcal{T}, \Sigma, B))$.

Therefore, by Definition 4, $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \models C' \sqsubseteq \exists r.D'$, and $\mathtt{UI}(\mathbb{G}_1, \mathbb{G}_2, \Sigma) \models C \sqsubseteq D_i$.

$2 \Rightarrow 3$**:** Observe that $\mathbb{G}_1, \mathbb{G}_2$ have $|\mathrm{sig}_C(\mathcal{T})|$ non-terminals and at most $2^{2 \cdot n} + |\mathrm{sig}_C(\mathcal{T})|$ outgoing transitions for each non-terminal, n the maximal arity of $\sqcap$, each of which has at most n occurring non-terminals. Let $\mathtt{leaves}_i$ be the maximal number of non-terminals $\mathfrak{n} \notin \mathcal{N}^\Sigma$ occurring in a transition after step i and $\mathtt{tran}_i$ the maximal number of outgoing transitions for a non-terminal after step i. Then, $\mathtt{tran}_0 = 2^{2 \cdot n} + |\mathrm{sig}_C(\mathcal{T})|$ and $\mathtt{leaves}_0 = n$. Further, $\mathtt{leaves}_{i+1} = n \cdot \mathtt{leaves}_i$, i.e., $\mathtt{leaves}_i = n^{i+1}$. For each $\mathfrak{n} \notin \mathcal{N}^\Sigma$, there are at most $2^{2 \cdot n} + |\mathrm{sig}_C(\mathcal{T})|$ possible replacing transitions, therefore, for each $t \in R_i$, there are $(2^{2 \cdot n} + |\mathrm{sig}_C(\mathcal{T})|)^{\mathtt{leaves}_{i+1}}$ possibilities to replace all non-terminals $\mathfrak{n} \notin \mathcal{N}^\Sigma$ by the corresponding transitions from R_0. We obtain $\mathtt{tran}_{i+1} = \mathtt{tran}_i \cdot (2^{2 \cdot n} + |\mathrm{sig}_C(\mathcal{T})|)^{\mathtt{leaves}_{i+1}}$, i.e., $\mathtt{tran}_i \leq (2^{2 \cdot n} + |\mathrm{sig}_C(\mathcal{T})|)^{i \cdot n^{i+2}}$. For $i = N$, we obtain $\mathtt{leaves}_i = n^N \in O(2^{2^{|\mathcal{T}|}})$ and $\mathtt{tran}_i \leq (2^{2 \cdot n} + |\mathrm{sig}_C(\mathcal{T})|)^{(N) \cdot n^{N+2}} \in O(2^{2^{2^{|\mathcal{T}|}}})$.

These complexity results correspond to the size and number of axioms in Example 2. □

7 Summary and Future Work

In this paper, we provide an approach to computing uniform interpolants of general $\mathcal{EL}$ terminologies based on proof theory and regular tree languages. Moreover, we show that, if a finite uniform $\mathcal{EL}$ interpolant exists, then there exists one of at most triple exponential size in terms of the original TBox, and that, in the worst-case, no shorter interpolant exists, thereby establishing the triple exponential tight bounds.

Due to the triple exponential blowup, algorithms for testing the appropriate size of uniform interpolants in addition to their existence would be of importance for applications in practice. While, in principle, expressing uniform interpolants in $\mathcal{EL}$ extended with fixpoint constructs [13] allows us to avoid both problems, the non-existence and the triple exponential blowup, for practical scenarios, reducing the forgotten signature in a reasonable way would be an interesting alternative, for instance, for applications as visualization of dependencies or ontology reuse.

Moreover, given the considerable effect of structure sharing elimination on the size of a TBox, it would be interesting to investigate, to what extent the structure sharing within existing large ontologies can be intensified in order to make reasoning more efficient.

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A Proof Theory

The structure of the grammars has been derived based on Proof Theory. The used Gentzen-style proof system shown below has been derived similarly to the proof system for Horn-$\mathcal{SHIQ}$ terminologies presented in [6]. In contrast to the proof system by Kazakov, which is complete for classification only and based on a normalization involving inverse roles (e.g., encoding all $\exists r.A \sqsubseteq B$ as $A \sqsubseteq \forall r^{-}.B$), the rules presented below fit our normal form and are complete for arbitrary $\mathcal{EL}$ GCIs.

$$\frac{}{C \sqsubseteq C}(\textsc{Ax}) \qquad \frac{}{C \sqsubseteq \top}(\textsc{AxTop})$$

$$\frac{D \sqsubseteq E}{C \sqcap D \sqsubseteq E}(\textsc{AndL})$$

$$\frac{C \sqsubseteq E \quad C \sqsubseteq D}{C \sqsubseteq D \sqcap E}(\textsc{AndR})$$

$$\frac{C \sqsubseteq D}{\exists r.C \sqsubseteq \exists r.D}(\textsc{Ex})$$

$$\frac{C \sqsubseteq E \quad E \sqsubseteq D}{C \sqsubseteq D}(\textsc{Cut})$$

Figure 2. Gentzen-style proof system for general $\mathcal{EL}$ terminologies.

Lemma 4 (Soundness and Completeness) *Let $\mathcal{T}$ be an arbitrary $\mathcal{EL}$ TBox, C, D $\mathcal{EL}$ concepts. Then $\mathcal{T} \models C \sqsubseteq D$, iff $\mathcal{T} \vdash C \sqsubseteq D$.*

Proof. While the soundness of the proof system (if-direction) can be easily checked for each rule, the proof of completeness is more sophisticated. In order to show the only-if-direction of the lemma, we construct a model $\mathcal{I}$ for $\mathcal{T}$ wherein *only* the GCIs derivable from $\mathcal{T}$ are valid. This model is constructed as follows:

- $\Delta^{\mathcal{I}}$ contains an element δ_C for every $\mathcal{EL}$ concept expression C
- $A^{\mathcal{I}} := \{\delta_C \in \Delta^{\mathcal{I}} | \mathcal{T} \vdash C \sqsubseteq A, \}$
- $r^{\mathcal{I}} := \{(\delta_C, \delta_D) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} | \mathcal{T} \vdash C \sqsubseteq \exists r.D, r \in \mathrm{sig}_R(\mathcal{T})\}$

We will show that the following claim holds for $\mathcal{I}$:
For all $\delta_E \in \Delta^{\mathcal{I}}$ and $\mathcal{EL}$ concepts F holds $\delta_E \in F^{\mathcal{I}}$ iff $\mathcal{T} \vdash E \sqsubseteq F$. *(*)*

This claim can be exploited in two ways: First, we use it to show that $\mathcal{I}$ is indeed a model of $\mathcal{T}$. Let $C \sqsubseteq D \in \mathcal{T}$ and consider an arbitrary $\delta_G \in \Delta^{\mathcal{I}}$ with $\delta_G \in C^{\mathcal{I}}$. Via (*) we obtain $\mathcal{T} \vdash G \sqsubseteq C$ on the other hand, $\mathcal{T} \vdash C \sqsubseteq D$ due to $C \sqsubseteq D \in \mathcal{T}$. Thus we can derive $\mathcal{T} \vdash G \sqsubseteq D$ via (Cut) and consequently, applying (*) again, we obtain $\delta_G \in D^{\mathcal{I}}$. Thereby modelhood of $\mathcal{I}$ wrt. $\mathcal{T}$ has been proven.

Second, we use (*) to show that $\mathcal{I}$ is a counter-model for all GCIs not derivable from $\mathcal{T}$ as follows: Assume $\mathcal{I} \models C \sqsubseteq D$ but $\mathcal{T} \nvdash C \sqsubseteq D$. Then $\Delta^{\mathcal{I}}$ contains the element δ_C. From $\mathcal{T} \vdash C \sqsubseteq C$ and (*) we derive $\delta_C \in C^{\mathcal{I}}$, from $\mathcal{T} \nvdash C \sqsubseteq D$ and (*) we obtain $\delta_C \notin D^{\mathcal{I}}$. Hence we get $C^{\mathcal{I}} \not\subseteq D^{\mathcal{I}}$ and therefore $\mathcal{I} \not\models C \sqsubseteq D$, a contradiction.

It remains to prove (*). This is done by an induction on the maximal nesting depth of the operators $\sqcap$ and $\exists$. There are two base cases:

- for $F = \top$, the claim trivially follows from (AxTop),
- for $F \in \mathrm{sig}_C(\mathcal{T})$, it is a direct consequence of the definition.

we now consider the cases where F is a complex concept expression

- for $F = C_1 \sqcap \ldots \sqcap C_n$, we note that $\delta_E \in F^{\mathcal{I}}$ exactly if $\delta_E \in C_i^{\mathcal{I}}$ for all $i \in \{1 \ldots n\}$. By induction hypothesis, this means $\mathcal{T} \vdash E \sqsubseteq C_i$ for all $i \in \{1 \ldots n\}$. Finally, observe that $\{E \sqsubseteq C_i \mid 1 \leq i \leq n\}$ and $E \sqsubseteq C_1 \sqcap \ldots \sqcap C_n$ can be mutually derived from each other: (for "$\vdash$" this is a straightforward consequence of (AndR), for "$\dashv$" note that we can derive $\emptyset \overset{\textsc{Ax}}{\vdash} C_i \sqsubseteq C_i \overset{\textsc{AndL}^*}{\vdash} C_1 \sqcap \ldots \sqcap C_n \sqsubseteq C_i$ whence together with $E \sqsubseteq C_1 \sqcap \ldots \sqcap C_n$ follows $E \sqsubseteq C_i$ by (Cut).
- for $F = \exists r.G$, we prove the two directions separately. First assuming $\delta_E \in F^{\mathcal{I}}$ we must find $(\delta_E, \delta_H) \in r^{\mathcal{I}}$ for some H with $\delta_H \in G^{\mathcal{I}}$. This implies both $\mathcal{T} \vdash E \sqsubseteq \exists r.H$ (by definition) and $\mathcal{T} \vdash H \sqsubseteq G$ (via the induction hypothesis). From the latter, we can deduce $\mathcal{T} \vdash \exists r.H \sqsubseteq \exists r.G$ by (Ex) and consequently $\mathcal{T} \vdash E \sqsubseteq \exists r.G$. For the other direction, note that by definition, $\mathcal{T} \vdash E \sqsubseteq \exists r.G$ implies $(\delta_E, \delta_G) \in r^{\mathcal{I}}$. On the other hand, we get $\mathcal{T} \vdash G \sqsubseteq G$ by (Ax) and therefore $\delta_G \in G^{\mathcal{I}}$ by the induction hypothesis which yields us $\delta_E \in F^{\mathcal{I}}$. □

B Proof of Lower Bound

Theorem 5 *There exists a sequence of $(\mathcal{T}_n)$ of $\mathcal{EL}$ TBoxes and a fixed signature Σ such that*

- *the size of $\mathcal{T}_n$ is upper-bounded by a polynomial in n and*
- *the size of the smallest uniform interpolant of $\mathcal{T}_n$ w.r.t. Σ is lower-bounded by $2^{(2^{(2^n-1)})}$.*

Proof Sketch. For n a natural number, let the $\mathcal{EL}$ TBox $\mathcal{T}_n$ be given by

$$A_1 \sqsubseteq \overline{X_0} \sqcap ... \sqcap \overline{X_{n-1}} \tag{8}$$

$$A_2 \sqsubseteq \overline{X_0} \sqcap ... \sqcap \overline{X_{n-1}} \tag{9}$$

$$\sqcap_{\sigma \in \{r,s\}} \exists \sigma.(\overline{X_i} \sqcap X_0 \sqcap ... \sqcap X_{i-1}) \sqsubseteq X_i \qquad i < n \tag{10}$$

$$\sqcap_{\sigma \in \{r,s\}} \exists \sigma.(X_i \sqcap X_0 \sqcap ... \sqcap X_{i-1}) \sqsubseteq \overline{X_i} \qquad i < n \tag{11}$$

$$\sqcap_{\sigma \in \{r,s\}} \exists \sigma.(\overline{X_i} \sqcap \overline{X_j}) \sqsubseteq \overline{X_i} \quad j < i < n \tag{12}$$

$$\sqcap_{\sigma \in \{r,s\}} \exists \sigma.(X_i \sqcap \overline{X_j}) \sqsubseteq X_i \quad j < i < n \tag{13}$$

$$X_0 \sqcap ... \sqcap X_{n-1} \sqsubseteq B \tag{14}$$

Obviously, the size of $\mathcal{T}_n$ is polynomially bounded by n. We now consider sets $\mathcal{C}_k$ of concept descriptions inductively defined by $\mathcal{C}_0 = \{A_1, A_2\}$ and $\mathcal{C}_{k+1} = \{\exists r.C_1 \sqcap \exists s.C_2 \mid C_1, C_2 \in \mathcal{C}_k\}$. We find that $|\mathcal{C}_{k+1}| = |\mathcal{C}_k|^2$ and consequently $|\mathcal{C}_k| = 2^{(2^k)}$. Thus, the set $\mathcal{C}_{2^n-1}$ contains triply exponentially many different concepts, each of which is doubly exponential in the size of $\mathcal{T}_n$.

Obviously, for any k, every concept description from $\mathcal{C}_k$ uses only signature elements from A_1, A_2, r, s.

It is rather straightforward to check that $\mathcal{T}_n \models C \sqsubseteq B$ holds for each concept $C \in \mathcal{C}_{2^n-1}$: by induction on k, we can show that for any $C \in \mathcal{C}_k$ with $k < 2^n$ holds $\mathcal{T}_n \models C \sqsubseteq Y_0^k \sqcap \ldots \sqcap Y_{n-1}^k$ with

$$Y_i^k = \begin{cases} X_i \text{ if } \lfloor \frac{k}{2^i} \rfloor \mathsf{mod}\ 2 = 1 \\ \overline{X_i} \text{ if } \lfloor \frac{k}{2^i} \rfloor \mathsf{mod}\ 2 = 0 \end{cases},$$

i.e., Y_i^k indicates the ith bit of the number k in binary encoding. Then, $C \sqsubseteq B$ follows via the last axiom of $\mathcal{T}_n$.

Toward the claimed triple-exponential lower bound, we now show that every uniform interpolant of $\mathcal{T}_n$ for $\Sigma = \{A_1, A_2, B, r, s\}$ must contain for each $C \in \mathcal{C}_{2^n-1}$ a GCI of the form $C \sqsubseteq B'$ with $B' = B$ or $B' = B \sqcap F$ for some F (where we consider structural variants –

i.e., concept expressions which are equivalent w.r.t. the empty knowledge base – as syntactically equal). Toward a contradiction, we assume that this is not the case, i.e., there is a uniform interpolant $\mathcal{T}'$ and a $C \in \mathcal{C}_{2^n-1}$ where $C \sqsubseteq B' \notin \mathcal{T}'$ for any B' containing B as a conjunct.

Yet, since $C \sqsubseteq B$ must be a consequence of $\mathcal{T}'$, there must be a derivation of it. Looking at the derivation calculus from the last section, the last derivation step must be (ANDL) or (CUT). We can exclude (ANDL) since neither $\exists r.C' \sqsubseteq B$ nor $\exists s.C' \sqsubseteq B$ is the consequence of $\mathcal{T}'$ for any $C' \in \mathcal{C}_{2^n-2}$ (which can be easily shown by providing appropriate witness models of $\mathcal{T}'$). Consequently, the last derivation step must be an application of (CUT), i.e., there must be a concept $E \neq C$ such that $\mathcal{T}' \models C \sqsubseteq E$ and $\mathcal{T}' \models E \sqsubseteq B$. Without loss of generality, we assume that we consider a derivation where the branch of the derivation branch for $C \sqsubseteq E$ has minimal depth.

We now distinguish two cases: either E contains B as a conjunct or not.

- First we assume $E = E' \sqcap B$, i.e. the CUT rule was used to derive $C \sqsubseteq B$ from $C \sqsubseteq E' \sqcap B$ and $E' \sqcap B \sqsubseteq B$. The former cannot be contained in $\mathcal{T}'$ by assumption, hence it must have been derived itself. Again, it cannot have been derived via (ANDL) for the same reasons as given above, which again leaves (CUT) as the only possible derivation rule for obtaining $C \sqsubseteq E' \sqcap B$. Thus, there must be some concept G with $\mathcal{T}' \models C \sqsubseteq G$ and $\mathcal{T}' \models G \sqsubseteq E' \sqcap B$. Once more, we distinguish two cases: either G contains B as a conjunct or not.
  - If G contains B as a conjunct, i.e., $G = G' \sqcap B$, the derivation of $C \sqsubseteq E$ was not depth-minimal since there is a better proof where $C \sqsubseteq B$ is derived from $C \sqsubseteq G' \sqcap B$ and $G' \sqcap B \sqsubseteq B$ via (CUT). Hence we have a contradiction.
  - If G does not contain B as a conjunct, the original derivation of $C \sqsubseteq E$ was not depth-minimal since we can construct a better one that derives $C \sqsubseteq B$ directly from $C \sqsubseteq G$ and $G \sqsubseteq B$ (the latter being derived from $G \sqsubseteq E' \sqcap B$ via (ANDR)).
- Now assume E does not contain B as a conjunct.
  We construct $(\Delta, \cdot^{\mathcal{I}})$, the "characteristic interpretation" of C as follows (ϵ denoting the empty word):
  - $\Delta = \{w \mid w \in \{r, s\}^*, \text{ length}(w) < 2^n\}$
  - We define an auxiliary function χ associating a concept expression to each domain element: we let $\chi(\epsilon) = C$ and for every $wr, ws \in \Delta$ with $\chi(w) = \exists r.C_1 \sqcap \exists s.C_2$, we let $\chi(wr) = C_1$ and $\chi(ws) = C_2$.
  - the concepts and roles are interpreted as follows:
    * $A_\iota^{\mathcal{I}} = \{w \mid \chi(w) = A_\iota\}$ for $\iota \in \{1, 2\}$
    * $B^{\mathcal{I}} = \{\epsilon\}$
    * $X_i^{\mathcal{I}} = \{w \mid \lfloor \frac{\text{length}(w)}{2^i} \rfloor \text{mod } 2 = 0\}$ for $i < n$
    * $\overline{X_i}^{\mathcal{I}} = \{w \mid \lfloor \frac{\text{length}(w)}{2^i} \rfloor \text{mod } 2 = 1\}$ for $i < n$
    * $r^{\mathcal{I}} = \{\langle w, wr\rangle \mid wr \in \Delta\}$
    * $s^{\mathcal{I}} = \{\langle w, ws\rangle \mid ws \in \Delta\}$

  It is straightforward to check that $\mathcal{I}$ is a model of $\mathcal{T}_n$ and that $\epsilon \in C^{\mathcal{I}}$. Consequently, due to our assumption, $\epsilon \in E^{\mathcal{I}}$ must hold. Yet then, by construction, E can only be a proper "structural superconcept" of C, i.e., $\emptyset \models C \sqsubseteq E$ and $\emptyset \not\models E \sqsubseteq C$ must hold. We now obtain $\widetilde{E}$ by enriching E as follows: recursively, for every subexpression G of E satisfying $\emptyset \models G \sqsubseteq C'$ for some $C' \in \mathcal{C}_k$ for some $k < 2^n$, we substitute G by $G \sqcap Y_0^k \sqcap \ldots \sqcap Y_{n-1}^k$. Then, $\widetilde{E}$ directly corresponds to a finite tree interpretation $\mathcal{I}'$ which is a model of $\mathcal{T}_n$ (following from structural induction on subexpressions of $\widetilde{E}$) and the root individual of which satisfies $\widetilde{E}$ but not C (by assumption). Yet, the root individual cannot satisfy any other concept expression C'' from $\mathcal{C}_{2^n-1} \setminus \{C\}$ either, since this, via $\emptyset \models E \sqsubseteq C''$, would imply $\emptyset \models C \sqsubseteq C''$ which is not the case (by induction on k one can show that there cannot be a homomorphism between the associated tree interpretations of any two distinct concepts from any $\mathcal{C}_k$). In particular, we note that the root individual of $\mathcal{I}'$ also does not satisfy B. Thus, we have found a model of $\mathcal{T}_n$ witnessing $\mathcal{T}_n \not\models E \sqsubseteq B$, contradicting our assumption that $\mathcal{T}' \models E \sqsubseteq B$.

□

C Proof of Lemma 2

Here, we prove a stronger version of Lemma 2 (the difference is the stronger statement [A4]), which is used only within the induction-based proof of this lemma.

Let $\mathcal{T}$ be a normalized $\mathcal{EL}$ TBox and C, D two $\mathcal{EL}$ concepts with $sig(C) \cup sig(D) \subseteq sig(\mathcal{T})$ such that $\mathcal{T} \models C \sqsubseteq D$. For any $A \in sig_C(\mathcal{T})$, let $\texttt{Pre}(A) = \{M \subseteq sig_C(\mathcal{T}) \mid \mathcal{T} \models \bigsqcap_{B_i \in M} B_i \sqsubseteq A\}$. W.l.o.g., assume that

$$C = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.E_k$$

for $A_j \in sig_C(\mathcal{T})$ and $r_k \in sig_R(\mathcal{T})$, E_k $\mathcal{EL}$ concepts with $sig(E_k) \subseteq sig(\mathcal{T})$ for $1 \leq k \leq m$. Then, for all conjuncts D_i of D, the following is true: If $D_i \in sig_C(\mathcal{T})$, there is a set $M \in \texttt{Pre}(D_i)$ of $sig_C(\mathcal{T})$ concepts such that for each element B of M holds at least one of the conditions [A1]-[A2]:

(A1) There is an A_j in C such that $A_j = B$.
(A2) There are r_k, E_k and there exists $B' \in sig_C(\mathcal{T})$ such that $\mathcal{T} \models E_k \sqsubseteq B'$ and $B \equiv \exists r_k.B' \in \mathcal{T}$.

If $D_i = \exists r'.D'$ for $r' \in sig_R(\mathcal{T})$ and D' an $\mathcal{EL}$ concept , at least one of the conditions [A3]-[A4] holds:

(A3) There are r_k, E_k such that $r_k = r'$ and $\mathcal{T} \models E_k \sqsubseteq D'$.
(A4) There is $B \in sig_C(\mathcal{T})$ such that $\mathcal{T} \models B \sqsubseteq \exists r'.D'$ and $\mathcal{T} \models C \sqsubseteq B$ and for $C \sqsubseteq B$ at least one of the conditions [A1]-[A2] holds.

Proof. We consider all rules, that could have been the last rule applied in order to obtain the above sequent and show by induction on the length of the proof that, in each case, the lemma holds. Rules AXTOP, AX are the basecase, since each proof begins with one of them.

$(C \bowtie D \in \mathcal{T})$ In the case that $C \sqsubseteq D \in \mathcal{T}$ or $C \equiv D \in \mathcal{T}$, the lemma holds due to the normalization. Axioms within $\mathcal{T}$ can have the following form:

- $C, D \in \text{sig}_C(\mathcal{T})$. In this case, $\{C\} \in \texttt{Pre}(D)$. Therefore, condition [A1] holds.
- $C \in \text{sig}_C(\mathcal{T}), D = D_1 \sqcap ... \sqcap D_m$ with $D_1, ..., D_m \in \text{sig}_C(\mathcal{T})$. In this case, for each D_i with $1 \leq i \leq m$ holds $\{C\} \in \texttt{Pre}(D_i)$. Therefore, condition [A1] holds for each D_i.

- $C \in \text{sig}_C(\mathcal{T}), D = \exists r'.D'$ with $D' \in \text{sig}_C(\mathcal{T})$. This case corresponds to the condition [A4].

(AXTOP) Since the conjunction is empty in case $D = \top$, the lemma holds.

(AX) Since $C = D$, for each D_i there is a conjunct C_i of C with $C_i = D_i$. If $D_i \in \text{sig}_C(\mathcal{T})$, condition [A1] of the lemma holds. Otherwise, [A3].

(EX) If EX was the last applied rule, then $D_i = \exists r_k.D'$ and $\mathcal{T} \vdash D_k \sqsubseteq D'$. Therefore, [A3] of the lemma holds.

(ANDL) Assume that $C' \sqcap C'' = C$ such that $C' \sqsubseteq D$ is the antecedent. By induction hypothesis, the lemma holds for $C' \sqsubseteq D$. Since all conjuncts of C' are also conjuncts of C, the lemma holds also for $C \sqsubseteq D$.

(ANDR) Assume that $D = D_1 \sqcap D_2$, therefore, $C \sqsubseteq D_1$ and $C \sqsubseteq D_2$ is the antecedent. By induction hypothesis, the lemma holds for both, $C \sqsubseteq D_1$ and $C \sqsubseteq D_2$. Since all conjuncts of D are from either D_1 or D_2, the lemma also holds for $C \sqsubseteq D$.

(CUT) By induction hypothesis, the lemma holds for both elements of the antecedent, $C \sqsubseteq C_1$ and $C_1 \sqsubseteq D$. W.l.o.g., assume that $C_1 = \bigsqcap_{1 \le p \le r} A_p \sqcap \bigsqcap_{1 \le s \le t} \exists r'_s.E'_s$.

1. Assume that $D_i \in \text{sig}_C(\mathcal{T})$. Then, there is $M_1 \in \texttt{Pre}(D_i)$ such that [A1] or [A2] holds for each $B_1 \in M_1$.

A1 Assume that there is A_p with $A_p = B_1$. Then, by induction hypothesis, for $C \sqsubseteq A_p$, there is $M_p \in \texttt{Pre}(A_p)$ such that [A1] or [A2] holds for each $B'_1 \in M_p$. Let $M_{\text{part}}(B_1) = M_p$ and $M_{1,A1} \subseteq M_1$ be the set of all such B_1. Then, let $M_{\text{new}} = M_1 \setminus M_{1,A1} \cup \bigcup\{M_{\text{part}}(B_1) \mid B_1 \in M_{1,A1}\}$.

A2 Assume that for B_1 there are r'_s, E'_s and there exists $B' \in \text{sig}_C(\mathcal{T})$ such that $\mathcal{T} \models E'_s \sqsubseteq B'$ and $B \equiv \exists r'_s.B' \in \mathcal{T}$. Then, for $C \sqsubseteq \exists r'_s.E'_s$ can hold [A3] or [A4].

-(A3) There are r_k, E_k such that $r_k = r'_s$ and $\mathcal{T} \models E_k \sqsubseteq E'_s$. Then [A2] holds for $C \sqsubseteq B_1$, since $\mathcal{T} \models E_k \sqsubseteq B'$ and $B \equiv \exists r_k.B' \in \mathcal{T}$.

-(A4) There is $B'' \in nct$ such that $\mathcal{T} \models B'' \sqsubseteq \exists r'_s.E'_s$, $\mathcal{T} \models C \sqsubseteq B''$ and there is a set $M'' \in \texttt{Pre}(B'')$ such that for each element B' of M'' holds at least one of the conditions [A1]-[A2] w.r.t. $C \sqsubseteq B'$. Let $M_{\text{part}}(B_1) = M''$ and $M_{1,A4} \subseteq M_1$ be the set of all such B_1. Then, let $M'_{\text{new}} = M_{\text{new}} \setminus M_{1,A4} \cup \bigcup\{M_{\text{part}}(B_1) \mid B_1 \in (M_{1,A4} \setminus M_{1,A1})\}$.

Clearly, $M'_{\text{new}} \in \texttt{Pre}(D_i)$ and [A1] or [A2] holds for each $B_1 \in M'_{\text{new}}$ w.r.t. $C \sqsubseteq B_1$, i.e., the lemma holds for $C \sqsubseteq D_i$.

2. Assume that $D_i = \exists r'.D'$. Then, [A3] or [A4] hold.

A3 There are r'_s, E'_s such that $r' = r'_s$ and $\mathcal{T} \models E'_s \sqsubseteq D'$. Then, for $C \sqsubseteq \exists r'_s.E'_s$ one of [A3], [A4] holds:

-(A3) There are r_k, E_k such that $r_k = r'_s$ and $\mathcal{T} \models E_k \sqsubseteq E'_s$. Then [A3] holds for $C \sqsubseteq D_i$, since $\mathcal{T} \models E_k \sqsubseteq D'$ and $r_k = r'$.

-(A4) There is $B'' \in nct$ such that $\mathcal{T} \models B'' \sqsubseteq \exists r'_s.E'_s$, $\mathcal{T} \models C \sqsubseteq B''$ and there is a set $M'' \in \texttt{Pre}(B'')$ of $\text{sig}_C(\mathcal{T})$ concepts such that for each element B' of M'' holds at least one of the conditions [A1]-[A2] w.r.t. $C \sqsubseteq B'$. Since $\mathcal{T} \models B'' \sqsubseteq D_i$, [A4] holds for $\mathcal{T} \models C \sqsubseteq D_i$.

A4 There is $B \in nct$ such that $\mathcal{T} \models B \sqsubseteq \exists r'.D'$, $\mathcal{T} \models C_1 \sqsubseteq B$ and there is a set $M' \in \texttt{Pre}(B)$ such that for each element B' of M holds at least one of the conditions [A1]-[A2] w.r.t. $C_1 \sqsubseteq B'$. Then, we have the same situation as above with two subsumptions $C \sqsubseteq C_1$ and $C_1 \sqsubseteq B$, where $B \in \text{sig}_C(\mathcal{T})$. Therefore, the argumentation is the same as above implying that the claim of the lemma holds for $C \sqsubseteq B$, i.e., there is $M_1 \in \texttt{Pre}(B)$ such that [A1] or [A2] holds for each $B_1 \in M_1$. Then, [A4] holds for $C \sqsubseteq D_i$. □

D Proofs for Section 5

D.1 Theorem 2

Let $\mathcal{T}$ be a normalized $\mathcal{EL}$ TBox, Σ a signature and $A \in sig_C(\mathcal{T})$.

1. *For each $t \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$, there is a concept C with $t_C = t$ and $sig(C) \subseteq \Sigma$ such that $\mathcal{T} \models C \sqsubseteq A$.*
2. *For each $t \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$, there is a concept C with $t_C = t$ and $sig(C) \subseteq \Sigma$ such that $\mathcal{T} \models A \sqsubseteq C$.*

Proof. It is easy to check in Definition 2 that the grammars derive only terms containing atomic concepts and roles from Σ, since $\mathfrak{n}_B \to B$ only if $B \in \Sigma$ and $\mathfrak{n}_B \to \exists r(t)$ only if $r \in \Sigma$. Therefore, for any $A \in \text{sig}_C(\mathcal{T})$ and any $t_C \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A)) \cup L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$ holds $\text{sig}(C) \subseteq \Sigma$.

1. Let t be a term such that $t \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$. We prove the theorem by induction on the maximal nesting depth of functions in t.
   - Assume that t is an atomic concept B. B can only be derived from $\mathfrak{n}_A$ by n empty transitions (GL2), and, once $\mathfrak{n}_B$ is reached, the rule (GL1). Let $B_1, ..., B_n$ be such that $\mathfrak{n}_A \to \mathfrak{n}_{B_1} \to ... \to \mathfrak{n}_{B_n} \to \mathfrak{n}_B$. Then, by Definition 2, for each pair B_i, B_{i+1} holds $\mathcal{T} \models B_i \sqsupseteq B_{i+1}$, for B_n, B holds $\mathcal{T} \models B_n \sqsupseteq B$ and for A, B_1 holds $\mathcal{T} \models A \sqsupseteq B_1$. It follows that also $\mathcal{T} \models A \sqsupseteq B$, while $t = t_B$.
   - Assume that $t = \exists r(t')$ for some term t'. Then, the derivation of t from $\mathfrak{n}_A$ starts with n empty transitions (GL2) such that $\mathfrak{n}_{B'}$ for some $B' \in \text{sig}_C(\mathcal{T})$ is reached, and a subsequent application of (GL4) such that $\mathfrak{n}_B$ for some $B \in \text{sig}_C(\mathcal{T})$ is reached. As argued above about the applications of empty transitions, $\mathcal{T} \models A \sqsupseteq B'$ holds. Moreover, By Definition 2 (GL4) holds $B' \equiv \exists r.B \in \mathcal{T}$, and, therefore, $\mathcal{T} \models A \sqsupseteq \exists r.B$. Let C' be a concept with $t' = t_{C'}$. Then, the theorem holds for C' and $\mathfrak{n}_B$ by induction hypothesis, i.e., $\mathcal{T} \models B \sqsupseteq C'$. Therefore, $\mathcal{T} \models A \sqsupseteq \exists r.C'$, while $t = t_{\exists r.C'}$.
   - Assume that $t = \sqcap(t_1, ..., t_n)$ for a set of terms $t_1, ..., t_n$. Then, the derivation of t from $\mathfrak{n}_A$ starts with n empty transitions (GL2) such that $\mathfrak{n}_{B'}$ for some $B' \in \text{sig}_C(\mathcal{T})$ is reached, and a subsequent application of (GL3) such that, for a set of concepts $B_i \in \text{sig}_C(\mathcal{T})$ with $1 \le i \le n$ and $t_i \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, \mathfrak{n}_{B_i}))$, $\mathfrak{n}_{B_i}$ is reached. As argued above about the applications of empty transitions, $\mathcal{T} \models A \sqsupseteq B'$ holds. Let C_i be a concept with $t_i = t_{C_i}$. By induction hypothesis, $\mathcal{T} \models B_i \sqsupseteq C_i$. By Definition 2, $\mathcal{T} \models B' \sqsupseteq B_1 \sqcap ... \sqcap B_n$. Therefore, $\mathcal{T} \models B' \sqsupseteq C_1 \sqcap ... \sqcap C_n$ and $\mathcal{T} \models A \sqsupseteq C_1 \sqcap ... \sqcap C_n$ with $t = t_{C_1 \sqcap ... \sqcap C_n}$.
2. The proof of soundness of $\mathbb{G}^{\sqsubseteq}(\mathcal{T}, \Sigma))$ can be done in the same manner. Let t be a term such that $t \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$. We prove the theorem by induction on the maximal nesting depth of functions in t.
   - Assume that t is an atomic concept B. B can only be derived from $\mathfrak{n}_A$ by n empty transitions (GR2), and, once $\mathfrak{n}_B$ is

reached, the rule (GR1). Let $B_1, ..., B_n$ be such that $\mathfrak{n}_A \to \mathfrak{n}_{B_1} \to ... \to \mathfrak{n}_{B_n} \to \mathfrak{n}_B$. Then, by Definition 2, for each pair B_i, B_{i+1} holds $\mathcal{T} \models B_i \sqsubseteq B_{i+1}$, for B_n, B holds $\mathcal{T} \models B_n \sqsubseteq B$ and for A, B_1 holds $\mathcal{T} \models A \sqsubseteq B_1$. It follows that also $\mathcal{T} \models A \sqsubseteq B$ with $t = t_B$.

- Assume that $t = \exists r(t')$ for some term t'. Then, the derivation of t from $\mathfrak{n}_A$ starts with n empty transitions (GR2) such that $\mathfrak{n}_{B'}$ for some $B' \in \mathrm{sig}_C(\mathcal{T})$ is reached, and a subsequent application of a non-empty transition (GR2) such that $\exists r.\mathfrak{n}_B$ for some $B \in \mathrm{sig}_C(\mathcal{T})$ is reached. As argued above about the applications of empty transitions, $\mathcal{T} \models A \sqsubseteq B'$ holds. Moreover, By Definition 2 holds $\mathcal{T} \models B' \sqsubseteq \exists r.B$, and, therefore, $\mathcal{T} \models A \sqsubseteq \exists r.B$. Let C' be a concept with $t' = t_{C'}$. By induction hypothesis, $\mathcal{T} \models B \sqsubseteq C'$. Therefore, $\mathcal{T} \models A \sqsubseteq \exists r.C'$ with $t = t_{\exists r.C'}$.
- Assume that $t = \sqcap(t_1, ..., t_n)$ for a set of terms $t_1, ..., t_n$. Then, the derivation of t from $\mathfrak{n}_A$ starts with n empty transitions (GR2) such that $\mathfrak{n}_{B'}$ for some $B' \in \mathrm{sig}_C(\mathcal{T})$ is reached, and a subsequent application of (GR2) such that, for a set of concepts $B_i \in \mathrm{sig}_C(\mathcal{T})$ with $1 \leq i \leq n$, we reach $\sqcap(\sigma(C_1), ..., \sigma(C_n))$ where for each i holds either $C_i = B_i$ and $t_i \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, \mathfrak{n}_{B_i}))$ or $C_i = \exists r.B_i$ and $tt'_i \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, \mathfrak{n}_{B_i}))$ for $t_i = \exists r.t'_i$. By induction hypothesis, for each B_i there is a concept C'_i with $t_{C'_i} = t_i$ in case $C_i = B_i$ and $t_{C'_i} = t'_i$, otherwise, such that $\mathcal{T} \models B_i \sqsubseteq C'_i$. Since, for each C_i, by Definition 2 holds $\mathcal{T} \models B' \sqsubseteq C_i$, we obtain a concept C' by replacing each B_i with C'_i such that $\mathcal{T} \models B' \sqsubseteq C'$, and $t_{C'} \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, \mathfrak{n}_{B'}))$. Therefore, also $\mathcal{T} \models A \sqsubseteq C'$, and $t_{C'} \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, \mathfrak{n}_A))$. $\square$

We start the proof of completness with a Lemma.

Lemma 5 *Let $\mathcal{T}$ be a normalized $\mathcal{EL}$ TBox, $A \in sig_C(\mathcal{T})$ and $r \in sig_R(\mathcal{T})$. Let C an $\mathcal{EL}$ concept such that $\mathcal{T} \models A \sqsubseteq \exists r.C$. Then, there are $B_1, B_2 \in sig_C(\mathcal{T})$ with $B_1 \equiv \exists r.B_2 \in \mathcal{T}$ such that $\mathcal{T} \models A \sqsubseteq B_1$, $\mathcal{T} \models B_2 \sqsubseteq C$.*

Proof. Lemma 16 in [10] states that for a general $\mathcal{EL}$ TBox $\mathcal{T}$ with $\mathcal{T} \models C_1 \sqsubseteq \exists r.C_2$, where C_1, C_2 are $\mathcal{EL}$-concepts one of the following holds:

- there is a conjunct $\exists r.C'$ of C_1 such that $\mathcal{T} \models C' \sqsubseteq C_2$;
- there is a subconcept $\exists r.C'$ of $\mathcal{T}$ such that $\mathcal{T} \models C_1 \sqsubseteq \exists r.C'$ and $\mathcal{T} \models C' \sqsubseteq C_2$;

The first condition does not hold in this lemma, since $A \in \mathrm{sig}_C(\mathcal{T})$. Moreover, since in our case $\mathcal{T}$ is normalized, for each subconcept $\exists r.C'$ of $\mathcal{T}$ containing an existential restriction holds: there is an atomic concept $B_2 \in \mathrm{sig}_C(\mathcal{T})$ such that $B_2 = C'$ and there is an axiom of the form $B_1 \equiv \exists r.B_2 \in \mathcal{T}$ with $B_1 \in \mathrm{sig}_C(\mathcal{T})$. Additionally, from the above Lemma 16 follows $\mathcal{T} \models A \sqsubseteq \exists r.B_2$ and $\mathcal{T} \models B_2 \sqsubseteq C$. Since $\mathcal{T} \models B_1 \equiv \exists r.B_2$, it follows that also $\mathcal{T} \models A \sqsubseteq B_1$. $\square$

We proceed with proving the two parts of Theorem 3. In what follows, we say that a concept C can be obtained from a concept C' by weakening, meaning that C can obtained from C' by adding arbitrary conjuncts to arbitrary subexpressions.

D.2 Theorem 3

Let $\mathcal{T}$ be a normalized $\mathcal{EL}$ TBox, Σ a signature and $A \in sig_C(\mathcal{T})$.

1. *For each C with $sig(C) \subseteq \Sigma$ such that $\mathcal{T} \models C \sqsubseteq A$ there is a concept C' such that C can be obtained from C' by adding arbitrary conjuncts to arbitrary subexpressions and $t_{C'} \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$.*
2. *For each D with $sig(D) \subseteq \Sigma$ such that $\mathcal{T} \models A \sqsubseteq D$ holds: $t_D \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$.*

Proof. Let $\mathcal{T}$ be a normalized $\mathcal{EL}$ TBox, Σ a signature and $A \in \mathrm{sig}_C(\mathcal{T})$. W.l.o.g., we can assume that there is a concept C with

$$C = \bigsqcap_{1 \leq j \leq n} A_j \sqcap \bigsqcap_{1 \leq k \leq m} \exists r_k.E_k$$

with $A_j \in \Sigma$ for $1 \leq j \leq n$, $r_k \in \Sigma$ for $1 \leq k \leq m$ and E_k with $1 \leq k \leq m$ a set of $\mathcal{EL}$ concepts such that $\mathrm{sig}(E_k) \subseteq \Sigma$. Further, w.l.o.g., we can assume that all A_j are pairwise different.

1. We show that, for each such general C with $\mathrm{sig}(C) \subseteq \Sigma$ and $\mathcal{T} \models C \sqsubseteq A$, there is a concept C' such that C can be obtained from C' by weakening and $t_{C'} \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$. We prove the claim by induction of the role depth of C.
   - Assume role depth = 0. Then C is a conjunction of atomic concepts, i.e., $m = 0$ and $C = \bigsqcap_{1 \leq j \leq n} A_j$. Then, by Lemma 2, there is a set $M' \in \mathtt{Pre}(A)$ of atomic concepts such that, for each $B \in M'$, there is an A_j with $A_j = B$. Therefore, each $B \in M'$ is in Σ. Let $C'_1 = \bigsqcap_{B \in M'} B$. Since $M' \subseteq \{A_1, ...A_n\}$, C can be obtained from C'_1 by weakening. By Definition 2 (GL3), there is a rule $\mathfrak{n}_A \to \sqcap(\mathfrak{n}_{B_1}, ..., \mathfrak{n}_{B_o})$ with $\{B_1, ..., B_o\} = M'$. Since each $B \in M'$ is in Σ, we obtain by (GL1) $\mathfrak{n}_B \to B$. Since our grammars operate on unordered trees, it follows that $\mathfrak{n}_A \to^+_{G^{\sqsupseteq}(\mathcal{T},\Sigma,A))} t_{C'_1}$, i.e., $t_{C'_1} \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$ for any order of conjuncts in C'_1. Therefore, the theorem holds with $C' = C'_1$.
   - Assume that the role depth is greater than 0. As in the case above, there is a set $M' \in \mathtt{Pre}(A)$ of atomic concepts such that, for each $B \in M'$, [A1] or [A2] holds. Let $M'_1 = M' \cap \{A_1, ...A_n\}$ and $M'_2 = M' \setminus M'_1$. Let $C'_1 = \bigsqcap_{B \in M'_1} B$, and $C'_2 = \bigsqcap_{1 \leq f \leq p} \exists r'_f.E'_f$ with $\{\exists r'_1.E'_1, ..., \exists r'_p.E'_p\} = \{\exists r.E \mid$ for one of $B \in M'_2$ holds [A2] such that there exists $B' \in \mathrm{sig}_C(\mathcal{T})$ with $\mathcal{T} \models E \sqsubseteq B'$ and $B \equiv \exists r.B' \in \mathcal{T}\}$. Clearly,$C$ can be obtained from $C'_1 \sqcap C'_2$ by weakening. By Definition 2 (GL3), there is a rule $\mathfrak{n}_A \to \sqcap(\mathfrak{n}_{B_1}, ..., \mathfrak{n}_{B_o})$ with $\{B_1, ..., B_o\} = M'$. Moreover, for all $B \in M'_1$ holds $\mathfrak{n}_B \to B$ and for all $B_f \in M'_2$, there is $\exists r'_f.E'_f$ such that there exists $B'_f \in \mathrm{sig}_C(\mathcal{T})$ with $\mathcal{T} \models E'_f \sqsubseteq B'_f$ and $B_f \equiv \exists r'_f.B'_f \in \mathcal{T}$. By Definition 2 (GL4), $\mathfrak{n}_{B_f} \to \exists r'_f(\mathfrak{n}_{B'_f})$. By induction hypothesis, there is a concept E''_f such that $\mathfrak{n}_{B'_f} \to^+_{G^{\sqsupseteq}(\mathcal{T},\Sigma,A))} t_{E''_f}$ and E'_f can be obtained from E''_f by weakening. Therefore, $\mathfrak{n}_{B_f} \to^+_{G^{\sqsupseteq}(\mathcal{T},\Sigma,A))} \exists r'_f(t_{E''_f})$ and $\exists r'_f.E'_f$ can be obtained from $\exists r'_f.E''_f$ by weakening. Let $C''' = C'_1 \sqcap \bigsqcap_{B_f \in M'_2} \exists r'_f.E''_f$. Then, C can be obtained from C''' by weakening. Since our grammars operate on unordered trees, we obtain $\mathfrak{n}_A \to^+_{G^{\sqsupseteq}(\mathcal{T},\Sigma,A))} t_{C'''}$, i.e., $t_{C'''} \in L(G^{\sqsupseteq}(\mathcal{T}, \Sigma, A))$ for any order of conjuncts. Therefore, the theorem holds with $C' = C'''$.
2. We proceed with showing that for each such general C with $\mathrm{sig}(C) \subseteq \Sigma$ and $\mathcal{T} \models A \sqsubseteq C$ holds: $t_C \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$. We prove the claim by induction of the role depth of C. For each A_j, we know that $\mathcal{T} \models A \sqsubseteq A_j$ and $A_j \in \Sigma$, i.e., $A_j \in \mathtt{Post}_{\mathrm{Base}}(A)$.

By Definition 2 (GR2) or (GR3), $\mathfrak{n}_{A_j} \rightarrow A_j$ for all A_j and $\mathfrak{n}_A \rightarrow \sqcap(\mathfrak{n}_{A_1}, ..., \mathfrak{n}_{A_n})$, and, therefore, $t_C \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$. Assume a role depth > 0. For each $\exists r_k.E_k$, it follows from Lemma 5 that there are $B_1, B_2 \in \mathrm{sig}_C(\mathcal{T})$ with $B_1 \equiv \exists r_k.B_2 \in \mathcal{T}$ such that $\mathcal{T} \models A \sqsubseteq B_1$, $\mathcal{T} \models B_2 \sqsubseteq E_k$. Since $r_k \in \Sigma$, follows that $\exists r_k.B_2 \in \mathtt{Post}_{\text{Base}}(A)$. Moreover, by induction hypothesis follows that $t_{E_k} \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, B_2))$. An application of (GR2) or (GR3) in Definition 2 yields $t_C \in L(G^{\sqsubseteq}(\mathcal{T}, \Sigma, A))$. $\square$