Revisiting Semantics for Epistemic Extensions of Description Logics

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Abstract. Epistemic extensions of description logics (DLs) have been introduced several years ago in order to enhance expressivity and querying capabilities of these logics by knowledge base introspection. We argue that unintended effects occur when imposing the semantics traditionally employed on the very expressive DLs that underly the OWL 1 and OWL 2 standards. Consequently, we suggest a revisited semantics that behaves more intuitive in these cases and coincides with the traditional semantics on less expressive DLs. Moreover, we introduce a way of answering epistemic queries to OWL knowledge bases by a reduction to standard OWL reasoning. We provide an implementation of our approach and present first evaluation results.

1 Introduction

Enrichment of query languages has been widely studied in knowledge base and database communities. Reiter [8] takes the approach of treating databases as first order theories, and queries as formulas of the language (of the database) extended by epistemic modal operator. Similar notions has been introduced in querying knowledge bases formulated in description logic(DLs)—well-known family of logic-based knowledge representation formalisms [3]. With such operators one can express questions not only about external world described in the knowledge base, but also the knowledge base itself i.e., about what the knowledge base knows. In [2], the language of the description logic $\mathcal{ALC}$ [9], has been extended by the epistemic operator $\mathbf{K}$. It has been shown how such enriched description logics can formalise non first-order aspects of frame-based systems, like *epistemic queries* and *procedural rules* as well as impose *integrity constraints*. In this work we mainly focus on epistemic querying. For a detail treatment of all this features, we refer the readers to [2].

In description logics, *concept descriptions* are the basis for expressing knowledge. These descriptions are constructed from concept names and roles using constructors available in the logic. Concept names denote classes of object in certain domain e.g., *Course, Graduate,* etc whereas roles are binary relation among the domain objects e.g., *teachers, enrolled,* etc. Different description logics allow for different set of constructors. Further, the syntax also allows for individual names (constants), which along with concept descriptions describe properties of the objects of a domain. Like first-order modal logic [10], the enrichment of the syntax of a description logic with the **K**-operator is quite straight forward (see Defintion 3), i.e., we allow the **K**-operator in front of each concept description and roles. For example, from the concept description $\forall hasChild.Doctor$, which represents people with only Doctor as their children, we can easily get the concept description

$$\forall \mathbf{K} hasChild.\mathbf{K} Doctor \tag{1}$$

The operator **K** is to be read as "knows". The intended meaning of **K** is to talk about the known objects of the domain, for example (1) represents the class of people whose all known children are known to be doctors. Thus it allows us to question the introspective aspects of a knowledge base. For example, in order to formalise the query "all known father which are not known to have daughters", we could do an instance retrieval w.r.t. the (epistemic) concept

$$C \equiv \mathbf{K}(Male \sqcap Parent) \sqcap \neg\exists \mathbf{K} hasChild.Female$$

The instances of C are all those known fathers who are not explicitly stated to be fathers of some daughter as well as those for whom there is no evidence (explicit or implicit) of their being father of some daughter. Consider the following knowledge base

$$\Sigma = \{Male(john), Parent(john), hasChild(john, mary)\}$$

Querying for the instances of C w.r.t. Σ results in $\{john\}$ as we don't have the assertion $\neg\exists hasChild.Female(john)$ in Σ neither we have any evidence of *john* being a father of a daughter, as it is not stated that *mary* is a female. Now considering the concept obtained by dropping **K** in C for querying will lead to an empty result. Hence, in the spirit of non-monotonicity, there are cases where more interesting results can be derived via epistemic querying than in conventional sense.

For the semantics, one has to consider how to interpret the epistemic operator. In literature, the approach adopted for the semantics of the epistemic extensions of description logics is to extend the notion of the possible world models for propositional epistemic logic. The semantic obtained this way considers each standard DL interpretations as a possible world and the accessibility relation as an equivalence relation among these interpretations, thus allowing both for positive introspection as well as negative introspection – the transitivity of the relation allows for the positive introspection, i.e., Σ knows what it knows. Similarly, the transitivity and symmetry of the relation allows for the negative introspection [4].

Under this semantics, the concept descriptions with no occurrence of the **K**-operator are interpreted as in standard DL semantics by assigning a subset of the domain of the interpretation to the concept descriptions, a binary relation over the domain to the roles and an element of the domain to the constants, and interpret the constructors accordingly. As in epistemic logic, the interpretation of the **K**-operator requires one to consider all the accessible worlds from the current world. In other words, the interpretation of a concept of the form $\mathbf{K}C$ is given by the set of all objects which belong to the class specified by concept C in every accessible interpretation. When providing the semantics of an epistemic extension of a DL, certain issues arise like the assumption about the domains of the interpretations, the relationship between the domain of two interrelated (via accessibility relation) interpretations, the interpretation of constants (e.g., *john* in the above example) etc. The assumptions made in common are usually the *Comman Domain Assumption* requiring that the domain of each interpretation in such semantics to share a common infinite domain and the *Rigid Term Assumption* requiring each individual to be interpreted rigidly. In the upcoming sections, we will see that the notion of knowledge base consistency and other reasoning related notions are quite dependent upon these two assumptions, and restrict us to a certain set of constructors in the language of the knowledge bases. Allowing for a richer knowledge base language, can cause severe problems regarding the consistency of the knowledge bases. Thereby, we propose a new semantics which does not enforce any of the assumptions. The rest of the report is organised in the following way. We start by presenting some preliminaries in Section 2, by providing syntax and semantics of the description logic $\mathcal{SROIQ}$. It is not only one of the most expressive DLs but also the underlying logic of the web ontology language OWL [7]. In 2.2 of this section, we present the **K**-extension of $\mathcal{SROIQ}$, which we call $\mathcal{SROIQK}$, by providing its syntax and semantics. Note that all the notions presented here can easily be restricted to any fragment of $\mathcal{SROIQ}$. We also introduce the notion of epistemic entailment. We then, present a technique for checking such epistemic entailment using standard reasoners. This as well as the formal proof of the correctness of the technique is presented in Section 3. In Subsection 4, we discuss why we need to restrict the language of the knowledge base to $\mathcal{SRIQ} \setminus U$[1] (a fragment of $\mathcal{SROIQ}$ without nominals and the universal role), which if not taken into consideration can lead to certain unavoidable problems. To allow for a richer language for the knowledge bases, we introduce a new semantics in Section ??. Like for the current semantics, in Section 6 we present a similar technique for checking epistemic entailment under the new semantics and also provide the formal correctness. In the same section, we establish the compatibility of both the semantics for $\mathcal{SRIQ}$ knowledge bases so far the epistemic entailment is concerned. Once all the theoretical details are provided, we realise our technique by providing a system for checking epistemic entailment. With this regard, in Section 7 we present a system and discuss several implementation issues involved as well observation made during some early tests. Finally we conclude in Section 8 and identify some future works.

2 Preliminaries

In this section, we present an introduction to the description logic $\mathcal{SROIQ}$ [5] and its extension with the epistemic operator K.

2.1 Description Logics $\mathcal{SROIQ}$

We start by presenting the syntax and semantics of $\mathcal{SROIQ}$. It is an extension of $\mathcal{ALC}$ [9] by inverse roles($\mathcal{I}$), role hierarchies($\mathcal{H}$), nominals($\mathcal{O}$) and qualifying number restrictions($\mathcal{Q}$). Besides it also allows for several role constructs and axioms.

Definition 1. For the signature of $\mathcal{SROIQ}$ we have countably infinite disjoint sets N_C, N_R and N_I of *concept names*, *role names* and *individual names* respectively. Further the set N_R is partitioned into two sets namely, $\mathbf{R_s}$ and $\mathbf{R_n}$ of *simple* and *non-simple* roles respectively. The set $\mathbf{R}$ *of $\mathcal{SROIQ}$ roles* is

$$\mathbf{R} := U \mid N_R \mid N_R^-$$

where U is a special role called the *universal role*. Further, we define a function Inv on roles such that $\mathsf{Inv}(R) = R^-$ if R is a role name, $\mathsf{Inv}(R) = S$ if $R = S^-$ and $\mathsf{Inv}(U) := U$.

The set of $\mathcal{SROIQ}$ concepts (or simply concepts) is the smallest set satisfying the following properties:

[1] From now onward by a $\mathcal{SRIQ}$ knowledge base we mean knowledge base without use of the universal role U.

– every concept name $A \in N_C$ is a concept;
– $\top$(top) and $\bot$ (bottom) are concepts;
– if C, D are concepts, R is a role, S is a simple role, $a_1, \ldots, a_n$ are individual names and n a non-negative integer then following are concepts:

$$
\begin{array}{rl}
\neg C & \text{(negation)} \\
\exists S.\mathsf{Self} & \text{(self)} \\
C \sqcap D & \text{(conjunction)} \\
C \sqcup D & \text{(disjunction)} \\
\forall R.C & \text{(universal quantification)} \\
\exists R.C & \text{(existential quantification)} \\
\leq nS.C & \text{(at least number restriction)} \\
\geq nS.C & \text{(at most number restriction)} \\
\{a_1, \ldots, a_n\} & \text{(nominals / one-of)}
\end{array}
$$

An *RBox axiom* is an expression of one the following forms:

1. $R_1 \circ \cdots \circ R_n \sqsubseteq R$ where $R_1, \ldots, R_n, R \in \mathbf{R}$ and if $n = 1$ and $R_1 \in \mathbf{R_s}$ then $R \notin \mathbf{R_n}$,
2. $\mathsf{Ref}(R)$ (reflexivity), $\mathsf{Tra}(R)$ (transitivity), $\mathsf{Irr}(R)$ (irreflexivity), $\mathsf{Dis}(R, R')$ (role disjointness), $\mathsf{Sym}(R)$ (Symmetry), $\mathsf{Asy}(R)$ (Asymmetry) with $R, R \in \mathbf{R}$.

RBox axioms of the first form i.e., $R_1 \circ \cdots \circ R_n \sqsubseteq R$ are called *role inclusion axioms* (RIAs). An RIA is *complex* if $n > 1$. Whereas the RBox axioms of the second form e.g., $\mathsf{Ref}(R)$, are called *role characteristics.* A $\mathcal{SROIQ}$ RBox $\mathcal{R}$ is a finite set of RBox axioms such that the following conditions are satisfied:[2]

– there is a strict (irreflexive) partial order $\prec$ on $\mathbf{R}$ such that
  - for $R \in \{S, \mathsf{Inv}(S)\}$, we have that $S \prec R$ iff $\mathsf{Inv}(S) \prec R$ and
  - every RIA is of the form $R \circ R \sqsubseteq R$, $\mathsf{Inv(R)} \sqsubseteq R$, $R_1 \circ \ldots R_n \sqsubseteq R$, $R \circ R_1 \circ \cdots \circ R_n \sqsubseteq R$ or $R_1 \circ \cdots \circ R_n \circ R \sqsubseteq R$ where $R, R_1 \ldots, R_n \in \mathbf{R}$ and $R_i \prec R$ for $1 \leq i \leq n$.
– any role characteristic of the form $\mathsf{Irr}(S)$, $\mathsf{Dis}(S, S')$ or $\mathsf{Asy}(S)$ is such that $S, S' \in \mathbf{R_s}$ i.e., we allow only for simple role in these role characteristics.

[2] These conditions are enforced to avoid cycles in the RBoxes, which, if not taken care, would lead to undecidability. We usually call an RBox to be *regular* because of the first condition.

A *general concept inclusion axiom* (GCI) is an expression of the form $C \sqsubseteq D$, where C and D are $\mathcal{SROIQ}$ concepts. A *TBox* is a finite set of GCIs.

An *ABox axiom* is of the form $C(a)$, $R(a,b)$, $a \doteq b$ or $a \neq b$ for the individual names a and b, a role R and a concept C. A *ABox* is a finite set of ABox axioms.

A $\mathcal{SROIQ}$ *knowledge base* is a tuple $(\mathcal{T}, \mathcal{R}, \mathcal{A})$ where $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{A}$ are $\mathcal{SROIQ}$ TBox, RBox and ABox respectively. ♢

To define the semantics of $\mathcal{SROIQ}$, we introduce the notion of interpretations.

Definition 2. A $\mathcal{SROIQ}$ interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is composed of a non-empty set $\Delta^{\mathcal{I}}$, called the *domain of* $\mathcal{I}$ and a *mapping function* $\cdot^{\mathcal{I}}$ such that:

- $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ for every concept name A;
- $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for every role name $R \in N_R$;
- $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ for every individual name a.

Further the universal role U is interpreted as a total relation on $\Delta^{\mathcal{I}}$ i.e., $U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The bottom concept $\bot$ and top concept $\top$ are interpreted by $\emptyset$ and $\Delta^{\mathcal{I}}$ respectively. Now the mapping $.^{\mathcal{I}}$ is extended to roles and concepts as follows:

$$
\begin{array}{rcl}
(R^-)^{\mathcal{I}} &=& \{(x,y) \mid (y,x) \in R^{\mathcal{I}}\} \\
(\neg C)^{\mathcal{I}} &=& \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(\exists S.\mathsf{Self})^{\mathcal{I}} &=& \{x \mid (x,x) \in S^{\mathcal{I}}\} \\
(C \sqcap D)^{\mathcal{I}} &=& C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &=& C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} &=& \{p_1 \in \Delta^{\mathcal{I}} \mid \forall p_2.(p_1,p_2) \in R^{\mathcal{I}} \rightarrow p_2 \in C^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} &=& \{p_1 \in \Delta^{\mathcal{I}} \mid \exists p_2.(p_1,p_2) \in R^{\mathcal{I}} \wedge p_2 \in C^{\mathcal{I}}\} \\
(\leq nS.C)^{\mathcal{I}} &=& \{p_1 \in \Delta^{\mathcal{I}} \mid \#\{p_2 \mid (p_1,p_2) \in S^{\mathcal{I}} \wedge p_2 \in C^{\mathcal{I}}\} \leq n\} \\
(\geq nS.C)^{\mathcal{I}} &=& \{p_1 \in \Delta^{\mathcal{I}} \mid \#\{p_2 \mid (p_1,p_2) \in S^{\mathcal{I}} \wedge p_2 \in C^{\mathcal{I}}\} \geq n\} \\
\{a_1, \ldots, a_n\}^{\mathcal{I}} &=& \{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\}
\end{array}
$$

♢

where C, D are concepts, R, S are roles, n is a non-negative integer and $\#M$ represents the cardinality of the set M.

Given an axiom α (TBox, RBox or ABox axiom), we say the an interpretation $\mathcal{I}$ satisfies α, written $\mathcal{I} \models \alpha$, if it satisfies the condition given in Table 1. Similarly $\mathcal{I}$ satisfies a TBox $\mathcal{T}$, written $\mathcal{I} \models \mathcal{T}$, if it satisfies all the axioms in $\mathcal{T}$. The satisfaction of an RBox and an ABox by an interpretation is defined in the same way. We say $\mathcal{I}$ satisfies a knowledge base $\Sigma = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ if it satisfies $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{A}$. We write $\mathcal{I} \models \Sigma$. We call $\mathcal{I}$ a model of Σ. A knowledge base is said to be *consistent* if it has a model.

We now present the extension of the DL $\mathcal{SROIQ}$ by the epistemic operator K. We call this extension $\mathcal{SROIQK}$.

Table 1. Semantics of $\mathcal{SROIQ}$ axioms

Axiom α	$\mathcal{I} \models \alpha$, if
$R_1 \circ \cdots \circ R_n \sqsubseteq R$	$R_1^{\mathcal{I}} \circ \cdots \circ R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
$\mathsf{Tra}(R)$	$R^{\mathcal{I}} \circ R^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
$\mathsf{Ref}(R)$	$(x,x) \in R^{\mathcal{I}}$ for all $x \in \Delta^{\mathcal{I}}$
$\mathsf{Irr}(S)$	$(x,x) \notin S^{\mathcal{I}}$ for all $x \in \Delta^{\mathcal{I}}$
$\mathsf{Dis}(S,T)$	$(x,y) \in S^{\mathcal{I}}$ implies $(x,y) \notin T^{\mathcal{I}}$ for all $x,y \in \Delta^{\mathcal{I}}$
$\mathsf{Sym}(S)$	$(x,y) \in S^{\mathcal{I}}$ implies $(y,x) \in S^{\mathcal{I}}$ for all $x,y \in \Delta^{\mathcal{I}}$
$\mathsf{Asy}(S)$	$(x,y) \in S^{\mathcal{I}}$ implies $(y,x) \notin S^{\mathcal{I}}$ for all $x,y \in \Delta^{\mathcal{I}}$
$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
$R(a,b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$
$a \doteq b$	$a^{\mathcal{I}} = b^{\mathcal{I}}$
$a \not\doteq b$	$a^{\mathcal{I}} \neq b^{\mathcal{I}}$

2.2 K-extensions of $\mathcal{SROIQ}$

The embedding of the epistemic operator K into the description logic $\mathcal{ALC}$ was first proposed in [1]. The logic obtained is called $\mathcal{ALCK}$. A similar approach has been taken in [2], which we follow in this work, although we extend the DL $\mathcal{SRIQ}$ rather than $\mathcal{ALC}$ i.e., we consider $\mathcal{SROIQ}$ as the basis DL and call its K-extension $\mathcal{SROIQK}$. In $\mathcal{SROIQK}$ we allow K in front of the concepts and roles. In the following we provide the formal syntax and semantics of such language where N_C, N_R, N_I, $\mathbf{R}$ are as in Definition 1.

Definition 3. A $\mathcal{SROIQK}$ role is defined as follows:

- every $R \in \mathbf{R}$ is a $\mathcal{SROIQK}$ role;
- if R is a $\mathcal{SROIQK}$ role then so are $\mathsf{K}R$ and R^-.

We call a $\mathcal{SROIQK}$ role an *epistemic role* if $\mathbf{K}$ occurs in it. An epistemic role is *simple* if it is of the form $\mathbf{K}S$ where S is a simple $\mathcal{SROIQ}$ role. Now $\mathcal{SROIQK}$ concepts are defined as follows:

- every $\mathcal{SROIQ}$ concept is an $\mathcal{SROIQ}$ concept;
- if C and D are $\mathcal{SROIQK}$ concepts, and S and R are $\mathcal{SROIQK}$ roles with S being simple, then the following are $\mathcal{SROIQK}$ concepts:

$$\mathbf{K}C \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \forall R.C \mid \exists R.C \mid \leq nS.C \mid \geq nS.C$$ ◊

The semantics of $\mathcal{SROIQK}$ is given as *possible world semantics* in terms of *epistemic interpretations*. Thereby following assumptions are made:

1. all interpretations are defined over a fixed infinite domain Δ (Common Domain Assumption);
2. for all interpretations, the mapping from individuals to domains elements is fixed: it is just the identity function (Rigid Term Assumption).

Definition 4. An epistemic interpretation for $\mathcal{SROIQK}$ is a pair $(\mathcal{I}, \mathcal{W})$ where $\mathcal{I}$ is a $\mathcal{SROIQ}$ interpretation and $\mathcal{W}$ is a set of $\mathcal{SROIQ}$ interpretations, where $\mathcal{I}$ and all of $\mathcal{W}$ have the same infinite domain Δ with $N_I \subset \Delta$. The interpretation function $\cdot^{\mathcal{I},\mathcal{W}}$ is then defined as follows:

$$\begin{aligned}
a^{\mathcal{I},\mathcal{W}} &= a \quad \text{for } a \in N_I \\
A^{\mathcal{I},\mathcal{W}} &= A^{\mathcal{I}} \quad \text{for } A \in N_C \\
R^{\mathcal{I},\mathcal{W}} &= R^{\mathcal{I}} \quad \text{for } R \in N_R \\
\top^{\mathcal{I},\mathcal{W}} &= \Delta \quad \text{(the domain of } \mathcal{I}) \\
\bot^{\mathcal{I},\mathcal{W}} &= \emptyset \\
(C \sqcap D)^{\mathcal{I},\mathcal{W}} &= C^{\mathcal{I},\mathcal{W}} \cap D^{\mathcal{I},\mathcal{W}} \\
(C \sqcup D)^{\mathcal{I},\mathcal{W}} &= C^{\mathcal{I},\mathcal{W}} \cup D^{\mathcal{I},\mathcal{W}} \\
(\neg C)^{\mathcal{I},\mathcal{W}} &= \Delta \setminus C^{\mathcal{I},\mathcal{W}} \\
(\forall R.C)^{\mathcal{I},\mathcal{W}} &= \{p_1 \in \Delta \mid \forall p_2.(p_1, p_2) \in R^{\mathcal{I},\mathcal{W}} \to p_2 \in C^{\mathcal{I},\mathcal{W}}\} \\
(\exists R.C)^{\mathcal{I},\mathcal{W}} &= \{p_1 \in \Delta \mid \exists p_2.(p_1, p_2) \in R^{\mathcal{I},\mathcal{W}} \wedge p_2 \in C^{\mathcal{I},\mathcal{W}}\} \\
(\leq nR.C)^{\mathcal{I},\mathcal{W}} &= \{d \mid \#\{e \in C^{\mathcal{I},\mathcal{W}} \mid (d, e) \in R^{\mathcal{I},\mathcal{W}}\} \leq n\} \\
(\geq nR.C)^{\mathcal{I},\mathcal{W}} &= \{d \mid \#\{e \in C^{\mathcal{I},\mathcal{W}} \mid (d, e) \in R^{\mathcal{I},\mathcal{W}}\} \geq n\} \\
(\mathbf{K}C)^{\mathcal{I},\mathcal{W}} &= \textstyle\bigcap_{\mathcal{J} \in \mathcal{W}} (C^{\mathcal{J},\mathcal{W}}) \\
(\mathbf{K}R)^{\mathcal{I},\mathcal{W}} &= \textstyle\bigcap_{\mathcal{J} \in \mathcal{W}} (R^{\mathcal{J},\mathcal{W}})
\end{aligned}$$

where C and D are $\mathcal{SROIQK}$ concepts and R is a $\mathcal{SROIQK}$ role. Further for an epistemic role $(\mathbf{K}R)^-$, we set $[(\mathbf{K}R)^-]^{\mathcal{I}} := (\mathbf{K}R^-)^{\mathcal{I}}$. It also follows from the semantics that the extension of both $\mathbf{KK}P$ and $\mathbf{K}P$, for a role P, are equal. Hence, from now onward, whenever we encounter a role of the form $\mathbf{K}P$, we safely assume that P is $\mathbf{K}$-free. ◊

From the above one can see that $\mathsf{K}C$ is interpreted as the set of objects that are in the interpretation of C under every interpretation in $\mathcal{W}$. Note that the rigid term assumption implies the unique name assumption (UNA) i.e., for any interpretation $\mathcal{I} \in \mathcal{W}$ and for any two distinct individual names a and b we have that $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.

The notions of GCI, assertion, role hierarchy, ABox, TBox and knowledge base, and their interpretations as defined in Definition 1 and 2 can be extended to that of $\mathcal{SROIQK}$ by allowing for $\mathcal{SROIQK}$ concepts and $\mathcal{SROIQK}$ roles in their definitions. One notational exception is that we use $\Vdash$ instead of $\models$ while expressing a satisfaction relation. For example, if an epistemic interpretation $(\mathcal{I}, \mathcal{W})$ satisfies a TBox $\mathcal{T}$, we write $(\mathcal{I}, \mathcal{W}) \Vdash \mathcal{T}$.

An *epistemic model* for a $\mathcal{SROIQK}$ knowledge base $\Psi = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ is a *maximal* non-empty set $\mathcal{W}$ of $\mathcal{SROIQ}$ interpretations such that $(\mathcal{I}, \mathcal{W})$ satisfies $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{A}$ for each $\mathcal{I} \in \mathcal{W}$. A $\mathcal{SROIQK}$ knowledge base Ψ is said to be *satisfiable* if it has an epistemic model. The knowledge base Ψ *entails* an axiom φ, written $\Psi \Vdash \varphi$, if for every epistemic model $\mathcal{W}$ of Ψ, we have that for every $\mathcal{I} \in \mathcal{W}$, the epistemic interpretation $(\mathcal{I}, \mathcal{W})$ satisfies φ. By definition every $\mathcal{SROIQ}$ knowledge base is an $\mathcal{SROIQK}$ knowledge base. Note that a given $\mathcal{SROIQ}$ knowledge base Σ has up to isomorphism only one unique epistemic model which is the set of all models of Σ having infinite domain and satisfying the unique name assumption. We denote this model by $\mathcal{M}(\Sigma)$.

Note that for the signature, we usually assume N_I to countably infinite set. Here, however, we replace it assuming N_I to be finite and containing all the individuals occurring in the knowledge base under consideration. This should not cause an technical difficulty as we can always extend N_I when required.

3 Deciding Entailment of Epistemic Axioms

We now present a method for deciding epistemic entailment based on techniques for non-epistemic standard reasoning. Given a knowledge base Σ, we consider the problem of checking if an epistemic axiom α is entailed by Σ. Like in [6], [8] etc, we distinguish between the *querying language* and the *modelling language*. More precisely, on one hand we restrict the language of Σ to $\mathcal{SRIQ}$, the DL obtained from $\mathcal{SROIQ}$ by disallowing nominals and the universal role. On the other hand, we allow for a richer language, namely, $\mathcal{SROIQK}$ for the query language i.e., epistemic axioms like α are formulated in $\mathcal{SROIQK}$. One could argue for a richer language like $\mathcal{SROIQ}$ for formalising Σ. We will justify the reason for restricting the use of nominals and the universal role in Section 4.

The basic idea behind our technique is to decide entailment of an axiom containing **K** operators is to disassemble the axiom and call a reasoner acquiring all the named individuals[3] contained in extensions for every sub-expression preceded by **K**. The results obtained is then used to rewrite the axiom into a **K**-free one by expressing the individual set as one-of concept. For example, we will see how a $\mathcal{SROIQK}$ concept is rewritten into an equivalent $\mathcal{SROIQ}$. This method of rewriting will be justified by providing a formal proof of its correctness. Nevertheless, certain problems arise as the consequence of the assumptions taken in current epistemic semantics. In the following we discuss these problems and ways to overcome them.

The rigid term assumption requires that an individual name to be interpreted rigidly i.e., under every interpretation, an individual is interpreted by the same element of the domain. This enforces the condition of interpreting two distinct individuals by different elements of the domain as $N_I \subset \Delta$ (see Definition 4). This condition is usually referred as the *Unique Name Assumption (UNA)*. Now as we reduce epistemic entailment to standard reasoning steps and since the standard DL semantics does not ensure UNA in general, we need to explicitly axiomatize this condition.

Definition 5. Given a $\mathcal{SRIQ}$ knowledge base Σ, we denote by Σ_{UNA} the knowledge base $\Sigma \cup \{a \not\approx b \mid a, b \in N_I, a \neq b\}$. $\diamondsuit$

Fact 6. *The set of models of Σ_{UNA} is exactly the set of those models of Σ that satisfy the unique name assumption.*

Note that since Σ has a unique epistemic interpretation $\mathcal{M}(\Sigma)$ and each $\mathcal{I} \in \mathcal{M}(\Sigma)$ satisfies UNA, it follows from the above fact that $\mathcal{I} \models \Sigma_{\mathrm{UNA}}$.

The common domain assumption requires the domain of an epistemic interpretation to be infinite. Nevertheless, the standard DL reasoners adheres to a semantics that allows for both finite and infinite models. Hence, in order to be able to deploy a standard reasoner for our purpose, we have

[3] By a *named individual* we mean an individual which is the interpretation of some individual name occurring in the knowledge base. Individuals lacking this property are referred to as *anonymous individuals*.

to show that considering only the infinite models suffices and one can easily drop the finite models without changing the consequences. This is possible for $\mathcal{SRIQ}$ knowledge bases i.e., for any finite model of a given $\mathcal{SRIQ}$ knowledge base, we can come up with an infinite model such that both behave in a similar way in terms of satisfaction of axioms. The following definition and lemma provide a concrete construction for this.

Definition 7. For any $\mathcal{SRIQ}$ interpretation $\mathcal{I}$, the *lifting* of $\mathcal{I}$ to ω is the interpretation $\mathcal{I}_\omega$ defined as follows:

- $\Delta^{\mathcal{I}_\omega} := \Delta^{\mathcal{I}} \times \mathbb{N}$,
- $a^{\mathcal{I}_\omega} := \langle a^{\mathcal{I}}, 0\rangle$ for every $a \in N_I$,
- $A^{\mathcal{I}_\omega} := \{\langle x, i\rangle \mid x \in A^{\mathcal{I}} \text{ and } i \in \mathbb{N}\}$ for each concept name $A \in N_C$,
- $r^{\mathcal{I}_\omega} := \{(\langle x, i\rangle, \langle x', i\rangle) \mid (x, x') \in r^{\mathcal{I}} \text{ and } i \in \mathbb{N}\}$ for every role name $r \in N_R$. ◊

Lemma 8. *For all $\langle x, i\rangle \in \Delta^{\mathcal{I}_\omega}$ and all $\mathcal{SRIQ}$ concepts C that $\langle x, i\rangle \in C^{\mathcal{I}_\omega}$ if and only if $x \in C^{\mathcal{I}}$.*

Proof. The proof is by the induction on the structure of C:

- For the atomic concept, $\top$ or $\bot$ it follows immediately from the definition of $\mathcal{I}_\omega$.
- Let $C = \neg D$. For any $x \in \Delta^{\mathcal{I}}$ we have that
  $x \in (\neg D)^{\mathcal{I}}$
  $\Leftrightarrow x \notin D^{\mathcal{I}}$
  $\Leftrightarrow \langle x, i\rangle \notin D^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$ (Induction)
  $\Leftrightarrow \langle x, i\rangle \in (\neg D)^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$.
- Let $C = C_1 \sqcap C_2$. For any $x \in \Delta^{\mathcal{I}}$ we have that
  $x \in (C_1 \sqcap C_2)^{\mathcal{I}}$
  $\Leftrightarrow x \in C_1^{\mathcal{I}}$ and $x \in C_2^{\mathcal{I}}$
  $\Leftrightarrow \langle x, i\rangle \in C_1^{\mathcal{I}_\omega}$ and $\langle x, i\rangle \in C_2^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$ (Induction)
  $\Leftrightarrow \langle x, i\rangle \in (C_1 \sqcap C_2)^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$.
- Let $C = \exists R.D$ for $R \in \mathbf{R}$. For any $x \in \Delta^{\mathcal{I}}$ we have that
  $x \in (\exists R.D)^{\mathcal{I}}$
  $\Leftrightarrow$ there is a $y \in \Delta^{\mathcal{I}}$ such that $(x, y) \in R^{\mathcal{I}}$ and $y \in D^{\mathcal{I}}$
  $\Leftrightarrow$ there is $\langle y, i\rangle \in \Delta^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$ with $(\langle x, i\rangle, \langle y, i\rangle) \in R^{\mathcal{I}_\omega}$ and $\langle y, i\rangle \in D^{\mathcal{I}_\omega}$ (Def 7 and Induction)
  $\Leftrightarrow \langle x, i\rangle \in (\exists R.D)^{\mathcal{I}_\omega}$
- The rest of the cases can be proved analogously.

Lemma 9. *Let Σ be a $\mathcal{SRIQ}$ knowledge base. For any interpretation $\mathcal{I}$ we have that*

$$\mathcal{I} \models \Sigma \text{ if and only if } \mathcal{I}_\omega \models \Sigma.$$

Proof. First we note that it follows immediately from the definition of $\mathcal{I}_\omega$ that for any $\mathcal{SRIQ}$-role $R \in \mathbf{R}$ and $(\langle x, i\rangle, \langle y, i'\rangle) \in \Delta^{\mathcal{I}_\omega}$ for $i, i' \in \mathbb{N}$ we have that $(\langle x, i\rangle, \langle y, i'\rangle) \in R^{\mathcal{I}_\omega}$ if an only if $(x, y) \in R^{\mathcal{I}}$ and $i = i'$ for an interpretation $\mathcal{I}$. Now for any RIA $R_1 \circ \ldots R_n \sqsubseteq R$ we have that:
$\mathcal{I} \models R_1 \circ \ldots R_n \sqsubseteq R$
$\Leftrightarrow \mathcal{I} \models R_1^{\mathcal{I}} \circ \ldots R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
$\Leftrightarrow$ for any $x_0, \ldots, x_n \in \Delta^{\mathcal{I}}$, whenever $(x_{i-1}, x_i) \in R_i^{\mathcal{I}}$ for $1 \leq i \leq n$ then $(x_0, x_n) \in R^{\mathcal{I}}$
$\Leftrightarrow$ for any $x_0, \ldots, x_n \in \Delta^{\mathcal{I}}$ and any $j \in \mathbb{N}$, whenever $(\langle x_{i-1}, j\rangle, \langle x_i, j\rangle) \in R_i^{\mathcal{I}_\omega}$ for $1 \leq i \leq n$ then $(\langle x_0, j\rangle, \langle x_n, j\rangle) \in R^{\mathcal{I}_\omega}$
$\Leftrightarrow \mathcal{I}_\omega \models R_1 \circ \ldots R_n \sqsubseteq R$.
The second last equivalence holds as $(x_{i-1}, x_i) \in R_i^{\mathcal{I}}$ for $1 \leq i \leq n$ and any non-negative integer j implies that $(\langle x_{i-1}, j\rangle, \langle x_i, j\rangle) \in R_i^{\mathcal{I}_\omega}$. Similary $(\langle x_{i-1}, j_{i-1}\rangle, \langle x_i, j_i\rangle) \in R^{\mathcal{I}_\omega}$ for $1 \leq i \leq n$ implies that $(x_{i-1}, x_i) \in R^{\mathcal{I}}$ and that all j_i, s are equal. And the same holds for the role R.

Similary, for any role characteristic $\mathsf{Ref}(R)$, we have that:
$\mathcal{I} \models \mathsf{Ref}(R)$
$\Leftrightarrow (x, x) \in R^{\mathcal{I}}$ for all $x \in \Delta^{\mathcal{I}}$
$\Leftrightarrow (\langle x, j\rangle, \langle x, j\rangle) \in R^{\mathcal{I}_\omega}$ for any $j \in \mathbb{N}$ and $x \in \Delta^{\mathcal{I}}$
$\Leftrightarrow (\langle x, j\rangle, \langle x, j\rangle) \in R^{\mathcal{I}_\omega}$ for any $\langle x, j\rangle \in \Delta^{\mathcal{I}_\omega}$ as $\Delta^{\mathcal{I}_\omega} = \Delta^{\mathcal{I}} \times \mathbb{N}$
$\Leftrightarrow \mathcal{I}_\omega \models \mathsf{Ref}(R)$.
In the same way, we can prove for any of the rest of the role characteristics that whenever $\mathcal{I}$ models it so does $\mathcal{I}_\omega$. Consequently we have that for any role hierarchy $\mathcal{R}$, $\mathcal{I} \models \mathcal{R}$ if and only if $\mathcal{I}_\omega \models \mathcal{R}$.

Now for any GCI $C \sqsubseteq D$ and for any interpretation $\mathcal{I}$, invoking Lemma 8 yields $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ if and only if $C^{\mathcal{I}_\omega} \subseteq D^{\mathcal{I}_\omega}$. Further for any TBox $\mathcal{T}$, $\mathcal{I} \models \mathcal{T}$ if and only if $\mathcal{I}_\omega \models \mathcal{T}$.

Finally for an ABox $\mathcal{A}$ we show that for each assertion in $\alpha \in \mathcal{A}$, $\mathcal{I} \models \alpha$ if and only if $\mathcal{I}_\omega \models \alpha$.

- α is of the form $C(a)$: Now for an interpretation $\mathcal{I}$ it follows from the definition of $\mathcal{I}_\omega$ that $a^{\mathcal{I}_\omega} = (a^{\mathcal{I}}, 0)$. As we have already shown that $a^{\mathcal{I}} \in C^{\mathcal{I}}$ if and only if $(a^{\mathcal{I}}, i) \in C^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$. Hence we get that $a^{\mathcal{I}} \in C^{\mathcal{I}}$ if and only if $(a^{\mathcal{I}}, 0) \in C^{\mathcal{I}_\omega}$.
- Analogously we can show an interpretation $\mathcal{I}$ satisfies an assertion if and only if $\mathcal{I}_\omega$ does so.

In our technique, given a knowledge base Σ, any axiom containing $\mathbf{K}$s is rewritten into a $\mathbf{K}$-free one by replacing the sub-expression preceded by $\mathbf{K}$ with a one-of concept containing all the (named) individual that are retrieved as instances of the sub-expression w.r.t. Σ. The justification for this is that only named element can be long to the extension of expression of the form $\mathbf{K}C$ for some concept C. To prove this we exploit certain symmetries on the model set $\mathcal{M}(\Sigma)$. The idea is that one can freely swap or permute anonymous individuals in a model of Σ without compromising its modelhood. To prove it formally, we first require the following notion.

Definition 10. Given an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, a set Δ with $N_I \subseteq \Delta$, and a bijection $\varphi : \Delta^{\mathcal{I}} \to \Delta$ with $\varphi(a^{\mathcal{I}}) = a$ for all $a \in N_I$, the *renaming* of $\mathcal{I}$ according to φ, denoted by $\varphi(\mathcal{I})$, is defined as the interpretation $(\Delta, \cdot^{\varphi(\mathcal{I})})$ with:

- $a^{\varphi(\mathcal{I})} = \varphi(a^{\mathcal{I}}) = a$ for every individual name a
- $A^{\varphi(\mathcal{I})} = \{\varphi(z) \mid z \in A^{\mathcal{I}}\}$ for every concept name A
- $P^{\varphi(\mathcal{I})} = \{(\varphi(z), \varphi(w)) \mid (z, w) \in P^{\mathcal{I}}\}$ for every role name P $\Diamond$

Lemma 11. *Let Σ be a $\mathcal{SRIQ}$ knowledge base and let $\mathcal{I}$ be a model of Σ with infinite domain. Then, every renaming $\varphi(\mathcal{I})$ of $\mathcal{I}$ satisfies $\varphi(\mathcal{I}) \in \mathcal{M}(\Sigma)$.*

Proof. By definition, the renaming satisfies the common domain and rigid term assumption. Modelhood w.r.t. Σ immediately follows from the isomorphism lemma of first-order interpretations [11] since $\mathcal{I}$ and $\varphi(\mathcal{I})$ are isomorphic and φ is an isomorphism from $\mathcal{I}$ to $\varphi(\mathcal{I})$. $\square$

Note that by semantics, $\mathbf{K}D$ represents all the individuals which are in the extension of D under every interpretation $\mathcal{I} \in \mathcal{M}(\Sigma)$ for a given knowledge base Σ. If D is not a universal concept, then any anonymous individual in the extension of D, under some model of Σ, can be swapped into the position of another individual not in the extension of D. Since the modelhood is preserved, this serves as a counter example. Hence, one can prove that $\mathbf{K}D$ contains merely named individual, provided D is not universal. Formally,

Lemma 12. *Let Σ be a $\mathcal{SRIQ}$ knowledge base. For any epistemic concept $C = \mathbf{K}D$ with $\Sigma_{\mathrm{UNA}} \not\models D \equiv \top$ and $x \in \Delta$, we have that $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ iff x is named such that there is an individual name $a \in N_I$ with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\mathrm{UNA}} \models D(a)$.*

Proof. " $\Rightarrow$ "
Suppose that $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$. It means that

$$x \in \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} D^{\mathcal{J}}$$

To the contrary, suppose that there is no $a \in N_I$ such that $a^{\mathcal{I},\mathcal{M}(\Sigma)} = x$ and $\Sigma_{\text{UNA}} \models D(a)$ i.e., x is an anonymous element. Since $\Sigma_{\text{UNA}} \not\models \top \equiv D$, there is a model $\mathcal{I}'$ of Σ_{UNA} such that $D^{\mathcal{I}'} \neq \Delta^{\mathcal{I}'}$. This implies that there is a $y \in \Delta^{\mathcal{I}'}$ such that $y \notin D^{\mathcal{I}'}$. Considering $\mathcal{I}'_\omega$, we can invoke Lemma 9 to ensure $\mathcal{I}'_\omega \models \Sigma_{\text{UNA}}$, moreover Lemma 8 guarantees $\langle y, 1\rangle \notin D^{\mathcal{I}'_\omega}$. On the other hand, by construction, $\langle y, 1\rangle$ is anonymous. Let $\varphi : \Delta^{\mathcal{I}'} \times \mathbb{N} \to \Delta$ be a bijection such that $\varphi(a^{\mathcal{I}'_\omega}) = a^{\mathcal{I}}$ for all $a \in N_I$ and $\varphi(\langle y, 1\rangle) = x$. Such a φ exists, as $|\Delta^{\mathcal{I}'} \times \mathbb{N}| = |\Delta|$ and $\mathcal{I}'_\omega$ satisfies the unique name assumption. By Lemma 11, we get that $\varphi(\mathcal{I}'_\omega) \in \mathcal{M}(\Sigma)$. By the choice of φ we get $x \notin D^{\varphi(\mathcal{I}'_\omega)}$ due to $\langle y, 1\rangle \notin D^{\mathcal{I}'_\omega}$ and the fact that φ is an isomorphism. In particular,

$$x \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} D^{\mathcal{J}}$$

which is a contradiction.
" $\Leftarrow$ "
Suppose there is $a \in N_I$ such that $a^{\mathcal{I},\mathcal{M}(\Sigma)} = x$ and $\Sigma_{\text{UNA}} \models D(a)$. This implies that for any $\mathcal{I} \in \mathcal{M}(\Sigma)$ we have that $x \in D^{\mathcal{I}}$ as each such $\mathcal{I}$ satisfies UNA. Hence we get that $x \in \mathbf{K}D^{\mathcal{I},\mathcal{M}(\Sigma)}$.

Similarly for epistemic roles, we can prove that only certain individuals can belong to their extensions. But we have to take care of the exceptional case of the universal role.

Claim 13. *Let Σ be a knowledge base. For the universal role U we have:*

$$\mathbf{K}U^{\mathcal{I},\mathcal{M}(\Sigma)} = U^{\mathcal{I},\mathcal{M}(\Sigma)}$$

The claim follows trivially as $U^{\mathcal{J}} = \Delta \times \Delta$ for any $\mathcal{J} \in \mathcal{M}(\Sigma)$. This means that $\bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} U^{\mathcal{J}} = \Delta \times \Delta$. Thus, as in the case of concepts, whenever an epistemic concept contains a role of the form $\mathbf{K}U$, it will be simply replaced by U. That, for $\mathcal{SRIQ}$ knowledge bases, no other role than U is universal (in all models) is straightforward and can be shown using the construction from Definition 7.

We now prove that the extension of every role preceded by K (except for the universal one), only contains individual which satisfy certain properties.

Lemma 14. *Let Σ be a $\mathcal{SRIQ}$ knowledge base. For any epistemic role $R = \mathsf{K}P$ with $P \neq U$, and $x, y \in \Delta$ we have that $(x, y) \in R^{\mathcal{I},\mathcal{M}(\Sigma)}$ iff at least one of the following holds:*

1. *there are individual names $a, b \in N_I$ such that $a^{\mathcal{I},\mathcal{M}(\Sigma)} = x$, $b^{\mathcal{I},\mathcal{M}(\Sigma)} = y$ and $\Sigma_{\text{UNA}} \models P(a, b)$.*
2. *$x = y$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$.*

Proof

" $\Leftarrow$ "

Depending on which case hold, we make the following case distinction:

- Suppose that $x = y$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$. As $\mathcal{M}(\Sigma)$ the epistemic model of Σ, therfore every interpretation in $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies the UNA and by Fact 6 we get that $\mathcal{J} \models \Sigma_{\text{UNA}}$. This means for every interpretation $\mathcal{J} \in \mathcal{M}(\Sigma)$ we have that $(x', x') \in P^{\mathcal{I},\mathcal{M}(\Sigma)}$ i.e.,

$$(x', x') \in \bigcap_{\mathcal{J}\in\mathcal{M}(\Sigma)} P^{\mathcal{J},\mathcal{M}(\Sigma)}$$

  for any $x' \in \Delta$. By semantics, therefore, $(x', x') \in \mathsf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ for any $x' \in \Delta$. In particular, we have that $(x, y) \in \mathsf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ as $x = y$.
- Suppose there are $a, b \in N_I$ with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$, $y = b^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models P(a, b)$. By assumption we have that $\Sigma_{\text{UNA}} \models P(a, b)$. Therefore, we have that $(x, y) \in P^{\mathcal{I}}$ for any interpretation $\mathcal{I} \in \mathcal{M}(\Sigma)$ as each such $\mathcal{I}$ satisfies UNA. Hence $(x, y) \in \mathsf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$.

" $\Rightarrow$ "

We first suppose that the second case of the lemma does not hold. Therefore, we have to show that there are a, b with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$, $y = b^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models P(a, b)$. To the contrary suppose that there is no such $a, b \in N_I$. We distinguish two cases.

- There are a, b with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $y = b^{\mathcal{I},\mathcal{M}(\Sigma)}$ but $\Sigma_{\text{UNA}} \not\models P(a, b)$. Now $\Sigma \not\models P(a, b)$ implies that there is an interpretation $\mathcal{I}'$ with $(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \notin P^{\mathcal{I}'}$. Considering $\mathcal{I}'_\omega$, we can invoke Lemma 9 to ensure $\mathcal{I}'_\omega \models \Sigma_{\text{UNA}}$ and by construction we also obtain $(a^{\mathcal{I}'_\omega}, b^{\mathcal{I}'_\omega}) \notin P^{\mathcal{I}'_\omega}$. Let $\varphi : \Delta^{\mathcal{I}'} \times \mathbb{N} \to \Delta$ be a bijection such that $\varphi(c^{\mathcal{I}'_\omega}) = c^{\mathcal{I}}$ for all $c \in N_I$.

Such a φ exists, as $|\Delta^{\mathcal{I}'} \times \mathbb{N}| = |\Delta|$ and ${\mathcal{I}'}_\omega$ satisfies the unique name assumption. By Lemma 11, we get that $\varphi({\mathcal{I}'}_\omega) \in \mathcal{M}(\Sigma)$. Moreover $(a^{\varphi(\mathcal{I}'_\omega)}, b^{\varphi(\mathcal{I}'_\omega)}) = (\varphi(a^{\mathcal{I}'_\omega}), \varphi(b^{\mathcal{I}'_\omega})) \notin P^{\varphi(\mathcal{I}'_\omega)}$. In particular,

$$(a^{\mathcal{I}}, b^{\mathcal{I}}) = (x, y) \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} P^{\mathcal{J}}$$

which is a contradiction.

– Assume at least one of x, y is anonymous. W.l.o.g. let x be anonymous, the other case follows by symmetry. Considering $\mathcal{I}_\omega$, we again have $\mathcal{I}_\omega \models \Sigma_{\text{UNA}}$ by Lemma 9. By construction, $\langle x, 1\rangle$ is anonymous and $(\langle x, 1\rangle, \langle y, 0\rangle) \notin P^{\mathcal{I}_\omega}$. Let $\varphi : \Delta^{\mathcal{I}} \times \mathbb{N} \to \Delta$ be a bijection such that $\varphi(\langle x, 1\rangle) = x$ and $\varphi(\langle y, 0\rangle) = y$. Such a φ exists, since $|\Delta^{\mathcal{I}} \times \mathbb{N}| = |\Delta|$ and $\mathcal{I}_\omega$ satisfies the unique name assumption. By Lemma 11, we get that $\varphi(\mathcal{I}_\omega) \in \mathcal{M}(\Sigma)$. Moreover $(\varphi(\langle x, 1\rangle), \varphi(\langle y, 0\rangle)) \notin P^{\varphi(\mathcal{I}_\omega)}$. In particular,

$$(x, y) \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} P^{\mathcal{J}}$$

which again is a contradiction.

Now we suppose that the first case does not hold. We have to show that $x = y$ and $\Sigma \models \top \sqsubseteq \exists P.\mathsf{Self}$. Again we assume to its contrary and make the following case distinction:

– $x \neq y$:
Now either both of x and y are named individuals but $\Sigma \not\models P(a, b)$ or at least one of them is anonymous. We can generate contradiction as above.

– $x = y$ and but $\Sigma_{\text{UNA}} \not\models \top \sqsubseteq \exists P.\mathsf{Self}$:
We have to distinguish two cases. First, suppose that x is a named individual i.e., there is $a \in N_I$ with $a^{\mathcal{I}} = x$. Now as $\Sigma_{\text{UNA}} \not\models P(a, a)$, this leads to contradiction as shown above.
Second, suppose that x is anonymous. Since every $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies UNA, therefore, it follows from Fact 6 that $\mathcal{J} \models \Sigma_{\text{UNA}}$ for every $\mathcal{J} \in \mathcal{M}(\Sigma)$. This along with the fact that $\Sigma_{\text{UNA}} \not\models \top \sqsubseteq \exists P.\mathsf{Self}$ implies that there is some $\mathcal{I}' \in \mathcal{M}(\Sigma)$ such that $(u, u) \notin P^{\mathcal{I}'}$ for some $u \in \Delta$. We define a bijection $\varphi : \Delta \to \Delta$ such that $\varphi(u) = x$. By Lemma 11, we get that $\varphi(\mathcal{I}') \in \mathcal{M}(\Sigma)$. Moreover $(\varphi(u), \varphi(u)) \notin P^{\varphi(\mathcal{I}')}$. In particular,

$$(\varphi(u), \varphi(u)) = (x, x) \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} P^{\mathcal{J}}$$

and therefore, by semantics, $(x, y) \notin \mathsf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ which is a contradiction. $\square$

Based on Lemma 12 and 14, for given a knowledge base Σ we now define a translation procedure that maps epistemic concept expression to non-epistemic ones which are equivalent in all models of Σ.

Definition 15. Given a $\mathcal{SRIQ}$ knowledge base Σ, we define the function Φ_Σ mapping $\mathcal{SROIQK}$ concept expressions to $\mathcal{SROIQ}$ concept expressions as follows (where we let $\{\} = \emptyset = \bot$)[4]:

$$\Phi_\Sigma : \begin{cases} C \mapsto C \quad \text{if } C \text{ is an atomic or one-of concept, } \top \text{ or } \bot; \\ \mathbf{K}D \mapsto \begin{cases} \top & \text{if } \Sigma_{\text{UNA}} \models \Phi_\Sigma(D) \equiv \top \\ \{a \in N_I \mid \Sigma_{\text{UNA}} \models \Phi_\Sigma(D)(a)\} & \text{otherwise} \end{cases} \\ \exists \mathsf{K}S.\mathsf{Self} \mapsto \begin{cases} \exists S.\mathsf{Self} & \text{if } \Sigma_{\text{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self} \\ \{a \in N_I \mid \Sigma_{\text{UNA}} \models S(a,a)\} & \text{otherwise} \end{cases} \\ C_1 \sqcap C_2 \mapsto \Phi_\Sigma(C_1) \sqcap \Phi_\Sigma(C_2) \\ C_1 \sqcup C_2 \mapsto \Phi_\Sigma(C_1) \sqcup \Phi_\Sigma(C_2) \\ \neg C \mapsto \neg \Phi_\Sigma(C) \\ \exists R.D \mapsto \exists R.\Phi_\Sigma(D) \quad \text{for non-epistemic role } R \\ \exists \mathsf{K}P.D \mapsto \bigsqcup_{a \in N_I} \{a\} \sqcap \exists P.(\{b \in N_I \mid \Sigma_{\text{UNA}} \models P(a,b)\} \sqcap \Phi_\Sigma(D)) \\ \qquad \sqcup \begin{cases} \Phi_\Sigma(D) & \text{if } \Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self} \\ \bot & \text{otherwise} \end{cases} \\ \forall R.D \mapsto \forall R.\Phi_\Sigma(D) \quad \text{for non-epistemic role } R; \\ \forall \mathsf{K}P.D \mapsto \neg \Phi_\Sigma(\exists \mathsf{K}P.\neg D) \\ \geqslant nS.D \mapsto \geqslant nS.\Phi_\Sigma(D) \quad \text{for non-epistemic role } S; \\ \geqslant n\mathsf{K}S.D \mapsto \begin{cases} \bigsqcup_{a \in N_I} \{a\} \sqcap \geqslant nP.(\{b \in N_I \mid \Sigma_{\text{UNA}} \models P(a,b)\} \sqcap \Phi_\Sigma(D)) & \text{if } n > 1 \\ \Phi_\Sigma(\exists \mathsf{K}P.D) & \text{otherwise} \end{cases} \\ \leqslant nS.D \mapsto \leqslant nS.\Phi_\Sigma(D) \quad \text{for non-epistemic role } S; \\ \leqslant n\mathsf{K}S.D \mapsto \neg \Phi_\Sigma(\geqslant (n+1)\mathsf{K}S.D) \\ \Xi \mathsf{K}U.D \mapsto \Xi U.\Phi_\Sigma(D) \quad \text{for } \Xi \in \{\forall, \exists, \geqslant n, \leqslant n\} \end{cases}$$

$\Diamond$

We now proceed toward the formal proof of the correctness of the presented translation procedure. In the following lemma, we show that for given knowledge base Σ and $\mathcal{SROIQK}$ concept C, the extension of C and $\Phi_\Sigma(C)$ agree under each model in $\mathcal{M}(\Sigma)$.

Lemma 16. *Let Σ be a $\mathcal{SRIQ}$ knowledge base, x be an element of Δ, and C be a $\mathcal{SROIQK}$ concept. Then for any interpretation $\mathcal{I} \in \mathcal{M}(\Sigma)$, we have that $C^{\mathcal{I},\mathcal{M}(\Sigma)} = (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$.*

Proof. It suffices to show that for any $x \in \Delta$, $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ exactly when $x \in \Phi_\Sigma(C)^{\mathcal{I},\mathcal{M}(\Sigma)}$. To show this we use induction on the structure of the

[4] W.l.o.g. we assume that in the definition of Φ_Σ, $n \geq 1$.

C. For the base case (C is an atomic concept) and the cases where $C = \top$ or $C = \bot$, the lemma follows immediately from the definition of Φ_Σ. For the cases, where $C = C_1 \sqcap C_2$, $C = C_1 \sqcup C_2$ or $C = \neg D$, it follows from the standard semantics and induction hypothesis. We focus on the rest of the cases in the following.

i. $C = \mathbf{K}D$ and $\Sigma_{\text{UNA}} \not\models D \equiv \top$:
By Lemma 12, $x \in (\mathbf{K}D)^{\mathcal{I},\mathcal{M}(\Sigma)}$ if and only if there is an $a \in N_I$ with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models D(a)$. This is equivalent to $x \in \{a \in N_I \mid \Sigma_{\text{UNA}} \models D(a)\}^{\mathcal{I},\mathcal{M}(\Sigma)}$ and hence, by definition of Φ_Σ, to $x \in [\Phi_\Sigma(\mathbf{K}D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

ii. $C = \mathbf{K}D$ and $\Sigma_{\text{UNA}} \models D \equiv \top$:
Note that it trivially holds that if $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ then $x \in (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$ as $\Phi_\Sigma(C) = \top$. Hence we just prove that whenever $x \in (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$ then $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ also. To contrary, suppose this is not the case i.e., $x \in (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$ but $x \notin C^{\mathcal{I},\mathcal{M}(\Sigma)}$. Hence, by definition, we get that

$$x \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} D^{\mathcal{J}}$$

Therefore, there is an interpretation $\mathcal{I}' \in \mathcal{M}(\Sigma)$ such that $x \notin D^{\mathcal{I}'}$. Since $\mathcal{M}(\Sigma)$ is the epistemic model of Σ, hence $\mathcal{I}' \in \mathcal{M}(\Sigma)$ respects the unique name assumption and therefore, $\mathcal{I}' \models \Sigma_{\text{UNA}}$ with $D^{\mathcal{I}'} \neq \Delta$. Hence $\Sigma_{\text{UNA}} \not\models D \equiv \top$, which is a contradiction.

iii. $C = \exists \mathbf{K}S.\mathsf{Self}$
"$\Rightarrow$"
We have to distinguish two cases.
First, we suppose that $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self}$, therefore by definition $\Phi_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \exists S.\mathsf{Self}$. Now $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that for each $\mathcal{J} \in \mathcal{M}(\Sigma)$, we have that $(x,x) \in S^{\mathcal{J},\mathcal{M}(\Sigma)}$. In particular, $(x,x) \in S^{\mathcal{I},\mathcal{M}(\Sigma)}$. Therefore, $x \in [\exists S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and hence $x \in [\Phi_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\mathcal{I},\mathcal{M}(\Sigma)}$.
Second, suppose that $\Sigma_{\text{UNA}} \not\models \top \sqsubseteq \exists S.\mathsf{Self}$. As $x \in [\exists \mathbf{K}S.\mathsf{Self}]$ implies that $(x,x) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$, by Lemma 14 there is $a \in N_I$ such that $a^{\mathcal{I}} = x$ and $\Sigma_{\text{UNA}} \models S(a,a)$ i.e., $a \in \{c \in N_I \mid \Sigma_{\text{UNA}} \models S(c,c)\}$ which immediately implies that $x = a^{\mathcal{I}} \in [\Phi_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\mathcal{I},\mathcal{M}(\Sigma)}$ as per definition of Φ_Σ.
"$\Leftarrow$"
Suppose that $\Phi_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \exists \mathbf{K}S.\mathsf{Self}$. Hence it is the case that $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self}$. Now as each model in $\mathcal{M}(\Sigma)$ satisfies UNA, by Fact 6, we have that $\mathcal{J} \models \Sigma_{\text{UNA}}$ and hence $\mathcal{J} \models \top \sqsubseteq \exists S.\mathsf{Self}$ for

each $\mathcal{J} \in \mathcal{M}(\Sigma)$ i.e., for every $u \in \Delta$, we have that $(u,u) \in S^{\mathcal{J},\mathcal{M}(\Sigma)}$. In other words, for every $u \in \Delta$, we have that $(u,u) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$. In particular, we have that $x \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ and therefore by semantics, $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

Assume that $\Phi_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \{c \in N_I \mid \Sigma_{\text{UNA}} \models S(c,c)\}$. Consequently, there is $a \in N_I$ with $a^{\mathcal{I}} = x$ and $\Sigma_{\text{UNA}} \models S(a,a)$ which by Lemma 14, implies that $(x,x) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$. Therefore, we get that $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

iv. $C = \exists P.D$ and P is a simple role:
By semantics, $x \in (\exists P.D)^{\mathcal{I},\mathcal{M}(\Sigma)}$ if and only if there is $y \in \Delta$ such that $(x,y) \in P^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $y \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $y \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Hence it is equivalent to $x \in (\Phi_\Sigma(KD))^{\mathcal{I},\mathcal{M}(\Sigma)}$.

v. $C = \exists \mathbf{K}P.D$:
"$\Rightarrow$"
$x \in [\exists \mathbf{K}P.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that there is some $y \in \Delta$ with $(x,y) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ such that $y \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $y \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. By Lemma 14, $(x,y) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that at least one of the following should hold.

- There are $a, b \in N_I$ with $a^{\mathcal{I}} = x$ and $b^{\mathcal{I}} = y$ such that $\Sigma_{\text{UNA}} \models P(a,b)$: Consequently we have that $y = b^{\mathcal{I}} \in [\{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a,c)\} \sqcap \Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Now as $\mathcal{M}(\Sigma)$ is an epistemic model, every interpretation in $\mathcal{M}(\Sigma)$ satisfies the UNA, and hence by Fact 6, for every $\mathcal{J} \in \mathcal{M}(\Sigma)$ we have that $\mathcal{J} \models \Sigma_{\text{UNA}}$. This along with $\Sigma_{\text{UNA}} \models P(a,b)$ implies that $(a^{\mathcal{I}}, b^{\mathcal{I}}) = (x,y) \in P^{\mathcal{I},\mathcal{M}(\Sigma)}$ and therefore, $x \in [\exists P.(\{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a,c)\} \sqcap \Phi_\Sigma(D))]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Hence, $x = a^{\mathcal{I}} \in [\{a\} \sqcap \{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a,c)\} \sqcap \Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, which by definition of Φ_Σ implies that $x \in [\Phi_\Sigma(\exists \mathbf{K}P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.
- $x = y$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$: As $y \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore it immediately follows from the definition that $x \in [\Phi_\Sigma(\exists \mathbf{K}P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

"$\Leftarrow$"
Suppose that $x \in [\Phi_\Sigma(\exists \mathbf{K}P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. This means that at least one of the following should hold.

- $x \in [\Phi_\Sigma(\exists \mathbf{K}P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$:
  It implies that there is an $a \in N_I$ such that $a^{\mathcal{I}} = x$ and $a^{\mathcal{I}} \in [\exists P.(\{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a,c)\} \sqcap \Phi_\Sigma(D))]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Consequently there is some $b \in N_I$ such that $b^{\mathcal{I}} \in [[\{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a,c)\} \sqcap \Phi_\Sigma(D)]]^{\mathcal{I},\mathcal{M}(\Sigma)}$ i.e., $\Sigma_{\text{UNA}} \models P(a,b)$ and $b^{\mathcal{I}} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $b^{\mathcal{I}} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$. By Lemma 14, $\Sigma_{\text{UNA}} \models P(a,b)$ implies that $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$. Therefore we get that $x = a^{\mathcal{I}} \in [\exists \mathbf{K}P.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

– $x \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$:
Note that each $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies UNA, therefore, $\mathcal{J} \models \Sigma_{\text{UNA}}$. This implies that $\mathcal{J} \models \top \sqsubseteq \exists P.\mathsf{Self}$. In other words, for every $u \in \Delta$, we have that $(u, u) \in P^{\mathcal{J}}$ for each $\mathcal{J} \in \mathcal{M}(\Sigma)$ and therefore, by semantics, we get that $(u, u) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$. In particular, $(x, x) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$. Now as $x \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, we get that $x \in [\exists \mathbf{K}P.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

vi. $C = \geqslant n\mathbf{K}S.D$:
"$\Rightarrow$"
Depending on n we distinguish the following cases.

– $n = 1$:
$x \in [\geqslant 1\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$ means that $x \in [\exists \mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Earlier we showed that this is the case iff $x \in [\Phi_\Sigma(\exists \mathbf{K}S.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and therefore by definition, $x \in [\Phi_\Sigma(\geqslant 1\mathbf{K}S.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.
– $n > 1$:
$x \in [\geqslant n\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that there are $y_1, \ldots, y_m$ with $m \geq n$ such that $(x, y_i) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $y_i \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$ for each $i \leq m$. By induction, $y_i \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ for each $i \leq m$. By Lemma 14, we have $a, b_1, \ldots, b_n \in N_I$ such that $a^{\mathcal{I}} = x$, $b_i^{\mathcal{I}} = y_i$ and $\Sigma_{\text{UNA}} \models S(a, b_i)$ for each $i \leq m$. Now as $m \geq n$ and $b_i^{\mathcal{I}} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ for $i \leq m$, it follows from the semantics that $x = a^{\mathcal{I}} \in [\geqslant nS.(\{c \in N_I \mid \Sigma_{\text{UNA}} \models S(a, c)\} \sqcap \Phi_\Sigma(D))]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Hence, using definition of Φ_Σ, we obtain that $x \in [\Phi_\Sigma(\geqslant n\mathbf{K}S.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ as $x \in \{a\}^{\mathcal{I},\mathcal{M}(\Sigma)}$.

"$\Leftarrow$"
Suppose that $n > 1$. Therefore,

$$\Phi_\Sigma(\geqslant n\mathbf{K}S.D) = \bigsqcup_{c \in N_I} \{c\} \sqcap \geqslant nS.(\{c' \in N_I \mid \Sigma_{\text{UNA}} \models S(c, c')\} \sqcap \Phi_\Sigma(D))$$

Now $x \in [\Phi_\Sigma(\geqslant n\mathbf{K}P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that there are $a, b_1, \ldots, b_m \in N_I$, for $m \geq n$, such $a^{\mathcal{I}} = x$, $\Sigma_{\text{UNA}} \models S(a, b_i)$ and $b_i^{\mathcal{I}} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ for each $i \leq m$. Since each $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies UNA, therefore, $\mathcal{J} \models \Sigma_{\text{UNA}}$ and hence we get that $(a^{\mathcal{J}}, b_i^{\mathcal{J}}) \in S^{\mathcal{J}}$ for each $\mathcal{J} \in \mathcal{M}(\Sigma)$. Hence, It follows from the semantics that $(a^{\mathcal{I}}, b_i^{\mathcal{I}}) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$ for each $i \leq m$. Now as $m \geq n$ and $b_i^{\mathcal{I}} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$(by induction), we get that $x \in [\geqslant n\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.
Now assume that $n = 1$. Hence, $\Phi_\Sigma(\geqslant n\mathbf{K}S.D) = \Phi_\Sigma(\exists \mathbf{K}S.D)$. Now for $x \in [\exists \mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$ at least one of the following holds:

– there is $a, b \in N_I$ with $a^{\mathcal{I}} = x$ such that $\Sigma_{\text{UNA}} \models S(a, b)$ and $b^{\mathcal{I}} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $b^{\mathcal{I}} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$. By Lemma 14, we get that $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$ which along with $b^{\mathcal{I}} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that $x = a^{\mathcal{I}} \in [\geqslant 1\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

- $x \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self}$. By Lemma 14, we get that $(x, x) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$. By induction we have that $x \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$ which immediately implies that $x \in [\geqslant 1\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

vii. The rest of the cases can be proved in a similar fashion.

Moreover Lemma 16 allows us to establish the result that the translation function Φ_Σ can be used to reduce the problem of entailment of $\mathcal{SROIQK}$ axioms by $\mathcal{SRIQ}$ knowledge bases to the problem of entailment of $\mathcal{SROIQ}$ axioms, formally put into the following theorem.

Theorem 17. *For a $\mathcal{SRIQ}$ knowledge base Σ, $\mathcal{SROIQK}$ concepts C, D and an individual a, the following hold:*

1. $\Sigma \Vdash C(a)$ *exactly if* $\Sigma_{\text{UNA}} \models \Phi_\Sigma(C)(a)$.
2. $\Sigma \Vdash C \sqsubseteq D$ *exactly if* $\Sigma_{\text{UNA}} \models \Phi_\Sigma(C) \sqsubseteq \Phi_\Sigma(D)$.

Proof. For the first case, we see that $\Sigma \Vdash C(a)$ is equivalent to $a^{\mathcal{I},\mathcal{M}(\Sigma)} \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ which by Lemma 14 is the case exactly if $a^{\mathcal{I},\mathcal{M}(\Sigma)} \in \Phi_\Sigma(C)^{\mathcal{I},\mathcal{M}(\Sigma)}$ for all $\mathcal{I} \in \mathcal{M}(\Sigma)$. Since $\Phi_\Sigma(C)$ does not contain any **K**s, this is equivalent to $a^{\mathcal{I}} \in \Phi_\Sigma(C)^{\mathcal{I}}$ and hence to $\mathcal{I} \models \Phi_\Sigma(C)(a)$ for all $\mathcal{I} \in \mathcal{M}(\Sigma)$. Now we can invoke Fact 6 and Lemma 9 to see that this is the case if and only if $\Sigma_{\text{UNA}} \models \Phi_\Sigma(C)(a)$. The second case is proven in exactly the same fashion. □

Theorem 17 justifies the use of a standard DL reasoner in answering epistemic queries. A straightforward observation is that the translation procedure according to the definition of Φ_Σ, may require deciding several classical entailment problems and hence involves many calls to the reasoner. Nevertheless, the number of reasoner calls is bounded by the number of $\mathbf{K}$s occurring in the query.

4 Semantical Problems Caused by Nominals and the Universal Role

In Section 2, we have mentioned that one of the basic assumptions that is made regarding the epistemic interpretations is the *common domain assumption*. This requires the commonality of the domain of interpretation in each world as well as enforces this domain to be infinite. However, there is no prima facie reason, why the domain that is described by a knowledge base should not be finite, yet finite models are excluded from the consideration entirely. In the previous section, we have seen that for the DL upto $\mathcal{SRIQ}$, this assumption is not a hurdle because of the fact that every finite model of a $\mathcal{SRIQ}$ knowledge base can be raised to an infinite one without compromising their behaviour (i.e. the two models cannot be distinguished by means of the underlying logic), as shown in Lemma 9. We have also seen how the current (epistemic) semantics seems feasible for epistemic entailment for $\mathcal{SRIQ}$ knowledge bases. However, once we allow for nominals or the universal role, we encounter epistemic inconsistency of certain knowledge bases. As an example the knowledge base containing the axioms $\top \sqsubseteq \{a, b, c\}$ or $\top \sqsubseteq \leqslant 3U.\top$ has only models with at most three elements. Consequently, according to the prevailing epistemic semantics, these axioms are epistemically unsatisfiable.

We believe that this phenomenon is not intended but rather a side effect of a semantics crafted for and probed against less expressive description logics, as it contradicts the intuition behind the $\mathbf{K}$ operator. To overcome such a problem, we present a refinement of the semantics in the next section.

5 Extended Semantics for $\mathcal{SROIQK}$

In this new semantics, we neither enforce the common domain assumption nor the rigid term assumption. Hence, the domain we consider in a possible world can be of arbitrary size, (non-empty essentially) composed of arbitrary elements and different individual names can stand for different elements in each possible world i.e., we interpret individual names non-rigidly. Note that under the current semantics, by the extension of an epistemic concept $\mathbf{K}C$ we mean all the elements which belong to the extension of C in every possible world. This justifies intersecting of the interpretation of C under each interpretation as in Definition 4. But in the new semantics as we don't enforce any of the assumptions i.e., CDA or RTA, interpreting $\mathbf{K}C$ in this manner leads to unsatisfiability. For example, in a world, we can interpret C by the set of individuals in way that none of these individual occurs in the extension of C in any other world. As intersecting all these extensions yields to an empty set, therefore, to unsatisfiability of the concept $\mathbf{K}C$. To overcome this problem, we propose the notion of *designators*. The idea here is to extend a standard interpretation $\mathcal{I}$ by a mapping from the set $N_I \cup \mathbb{N}$ to the domain $\Delta^{\mathcal{I}}$ of $\mathcal{I}$. To this end, we define the notion of an extended interpretation.

Definition 18. An extended $\mathcal{SROIQ}$ interpretation $\tilde{\mathcal{I}}$ is a tuple $(\Delta^{\tilde{\mathcal{I}}}, \cdot^{\tilde{\mathcal{I}}}, \varphi_{\tilde{\mathcal{I}}})$ such that

1. $(\Delta^{\tilde{\mathcal{I}}}, \cdot^{\tilde{\mathcal{I}}})$ is a standard $\mathcal{SROIQ}$ interpretation,

2. $\varphi_{\tilde{\mathcal{I}}}$ is a surjective function $\varphi_{\tilde{\mathcal{I}}} : N_I \cup \mathbb{N} \to \Delta^{\tilde{\mathcal{I}}}$, such that for all $a \in N_I$ we have that $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}}$.

We extend the definition of $\varphi_{\tilde{\mathcal{I}}}$ to subsets of $N_I \cup \mathbb{N}$. For a set S, $\varphi_{\tilde{\mathcal{I}}}(S) := \{\varphi_{\tilde{\mathcal{I}}}(t) \mid t \in S\}$. Similarly we extend $\varphi_{\tilde{\mathcal{I}}}$ to order pairs and set of order pairs on $N_I \cup \mathbb{N}$ as follows:

- $\varphi_{\tilde{\mathcal{I}}}((s,t)) := (\varphi_{\tilde{\mathcal{I}}}(s), \varphi_{\tilde{\mathcal{I}}}(t))$ for some ordered-pair $(s,t) \in (N_I \cup \mathbb{N})^2$.
- $\varphi_{\tilde{\mathcal{I}}}(T) := \{\varphi_{\tilde{\mathcal{I}}}((s,t)) \mid (s,t) \in T\}$ for some set $T \subseteq (N_I \cup \mathbb{N})^2$. ♢

We also define the inverse $\varphi_{\tilde{\mathcal{I}}}^{-1}$ of the mapping $\varphi_{\tilde{\mathcal{I}}}$ for an extended interpretation $\tilde{\mathcal{I}}$ as follows:

- $\varphi_{\tilde{\mathcal{I}}}^{-1}(x) := \{t \in N_I \cup \mathbb{N} \mid \varphi_{\tilde{\mathcal{I}}}(t) = x\}$ for every $x \in \Delta^{\tilde{\mathcal{I}}}$.
- $\varphi_{\tilde{\mathcal{I}}}^{-1}(E) := \{\varphi_{\tilde{\mathcal{I}}}^{-1}(x) \mid x \in E\}$ for $E \subseteq \Delta^{\tilde{\mathcal{I}}}$.
- $\varphi_{\tilde{\mathcal{I}}}^{-1}((x,y)) := \varphi_{\tilde{\mathcal{I}}}^{-1}(x) \times \varphi_{\tilde{\mathcal{I}}}^{-1}(y) = \{(x',y') \mid x' \in \varphi_{\tilde{\mathcal{I}}}^{-1}(x) \text{ and } y' \in \varphi_{\tilde{\mathcal{I}}}^{-1}(y)\}$ for any ordered-pair $(x,y) \in \Delta^{\tilde{\mathcal{I}}} \times \Delta^{\varphi_{\tilde{\mathcal{I}}}}$.
- $\varphi_{\tilde{\mathcal{I}}}^{-1}(H) := \bigcup_{(x,y)\in H} \varphi_{\tilde{\mathcal{I}}}^{-1}((x,y))$ for any $H \subseteq \Delta^{\tilde{\mathcal{I}}} \times \Delta^{\tilde{\mathcal{I}}}$.

Note that for any extended interpretation $\tilde{\mathcal{I}}$, the definition of $\varphi_{\tilde{\mathcal{I}}}$ guarantees that each individual name a is the designator of the interpretation of a under $\tilde{\mathcal{I}}$. For the rest of the elements of $\Delta^{\tilde{\mathcal{I}}}$, we use elements of $\mathbb{N}$ as their designators. As we noted in Section 3, the extension of any concept (role) of the form $\mathbf{K}C$ ($\mathbf{K}R$) is composed of named individuals (couple of named individuals) only. This is reason that we treat individual names differently. Based on the notion of extended interpretation we now provide a new semantics for $\mathcal{SROIQK}$.

Definition 19. (extended semantics for $\mathcal{SROIQK}$)

An *extended epistemic interpretation* for $\mathcal{SROIQK}$ is a pair $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}})$, where $\tilde{\mathcal{I}}$ is an extended $\mathcal{SROIQ}$ interpretation and $\tilde{\mathcal{W}}$ is a set of extended $\mathcal{SROIQ}$ interpretations. Similar to epistemic interpretations, we define an extended interpretation function $\cdot^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}$:

$$
\begin{aligned}
a^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= a^{\tilde{\mathcal{I}}} \quad \text{for } a \in N_I \\
A^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= A^{\tilde{\mathcal{I}}} \quad \text{for } A \in N_C \\
R^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= R^{\tilde{\mathcal{I}}} \quad \text{for } R \in N_R \\
\top^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \Delta^{\tilde{\mathcal{I}}} \quad \text{(the domain of } \tilde{\mathcal{I}}) \\
\bot^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \emptyset \\
(C \sqcap D)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \cap D^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \\
(C \sqcup D)^{\mathcal{I},\mathcal{W}} &= C^{\mathcal{I},\mathcal{W}} \cup D^{\mathcal{I},\mathcal{W}} \\
(\neg C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \Delta^{\tilde{\mathcal{I}}} \setminus C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \\
(\forall R.C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \{p_1 \in \Delta^{\tilde{\mathcal{I}}} \mid \forall p_2.(p_1, p_2) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \rightarrow p_2 \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}\} \\
(\exists R.C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \{p_1 \in \Delta^{\tilde{\mathcal{I}}} \mid \exists p_2.(p_1, p_2) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \wedge p_2 \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}\} \\
(\leq nR.C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \{d \mid \#\{e \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \mid (d, e) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}\} \leq n\} \\
(\geq nR.C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \{d \mid \#\{e \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \mid (d, e) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}\} \geq n\} \\
(\mathbf{K}C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \varphi_{\tilde{\mathcal{I}}} \left(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{W}}} \varphi_{\tilde{\mathcal{J}}}^{-1} \left(C^{\tilde{\mathcal{J}},\tilde{\mathcal{W}}} \right) \right) \\
(\mathbf{K}R)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \varphi_{\tilde{\mathcal{I}}} \left(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{W}}} \varphi_{\tilde{\mathcal{J}}}^{-1} \left(R^{\tilde{\mathcal{J}},\tilde{\mathcal{W}}} \right) \right)
\end{aligned}
$$

Again, for an epistemic role $(\mathbf{K}R)^-$, we set $[(\mathbf{K}R)^-]^{\tilde{\mathcal{J}},\tilde{\mathcal{W}}} := (\mathbf{K}R^-)^{\tilde{\mathcal{J}},\tilde{\mathcal{W}}}$. Like in Definition 4, $\mathbf{KK}R$ and $\mathbf{K}R$ are interpreted identically under an epistemic interpretation. Hence, it suffices to consider only epistemic roles of the form $\mathbf{K}R$, with R being non-epistemic. $\Diamond$

Note that unlike the epistemic interpretations we don't force the common domain assumption— the domains of $\tilde{\mathcal{I}}$ and that of every extended interpretation in $\tilde{\mathcal{W}}$ need not to be infinite nor identical. Similarly we don't enforce the rigid term assumption either. When interpreting a concept of the form $\mathbf{K}C$ (and similarly for the role), we consider the intersection of the set of the designators of the elements of the extensions of C under each interpretation. We then consider the elements of the domain of the current interpretation, for which these designators stand. In contrast to current semantics, an element x is in the extension of $\mathbf{K}C$ exactly when for every possible world, there is some element in the extension of C in that world, which has the same designator as that of x. This to some extend justifies the use of separate designators for the interpretations of the named individuals.

The semantics of GCI, assertion, role hierarchy, ABox, TBox, RBox and knowledge base under an extended epistemic interpretation can be defined in a straight forward way like in Definition 2. Here, instead $\models$ as the symbol of the satisfaction relation, we use the symbol $\Vdash_{\mathrm{e}}$, where

e indicates that the relation is w.r.t. the extended semantics. Like the epistemic models, we introduce the notion of an extended epistemic model of a knowledge base.

Definition 20. An *extended epistemic model* of a $\mathcal{SROIQK}$ knowledge base $\Psi = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ is a *maximal* non-empty set $\tilde{\mathcal{W}}$ of extended $\mathcal{SROIQ}$ interpretations such that $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}})$ satisfies $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{A}$ for each $\tilde{\mathcal{I}} \in \tilde{\mathcal{W}}$. A $\mathcal{SROIQK}$ knowledge base Ψ is *satisfiable* (under the extended semantics) if it has an extended epistemic model. Similarly the knowledge base Ψ *entails* an axiom α, written $\Psi \models_{e} \alpha$, if for every extended epistemic model $\tilde{\mathcal{W}}$ of Ψ, we have that for every $\tilde{\mathcal{I}} \in \tilde{\mathcal{W}}$, the extended epistemic interpretation $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}})$ satisfies α. Like in case of the current semantics, a standard DL-knowledge base Σ, one without any occurence of $\mathbf{K}$, admits a unique extended epistemic model, which is the set of all models of Σ extended by all possible surjective mappings that map individuals names and elements of $\mathbb{N}$ to the elements of their domain. We denote this model by $\tilde{\mathcal{M}}(\Sigma)$.

Note that when considering non-epistemic axioms, the notions of satisfaction under the extended semantics and under the standard semantics coincide. More precisely, given a standard interpretation $\mathcal{I}$, by $\mathcal{E}(\mathcal{I})$ we mean the set of all extended interpretation $\tilde{\mathcal{I}}$ such that $\Delta^{\tilde{\mathcal{I}}} = \Delta^{\mathcal{I}}$ and the mappings $\cdot^{\mathcal{I}}$ and $\cdot^{\tilde{\mathcal{I}}}$ are identical. In other words, $\mathcal{E}(\mathcal{I})$ represents all the extended interpretations obtained by augmented some mapping $\varphi_{\tilde{\mathcal{I}}} : N_I \cup \mathbb{N} \to \Delta^{\mathcal{I}}$ satisfying $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}}$ for each $a \in N_I$. Based on this notion, we have the following

Fact 21. *For any $\mathcal{SROIQ}$ knowledge base Σ, we have that*

$$\tilde{\mathcal{M}}(\Sigma) = \{\tilde{\mathcal{I}} \mid \tilde{\mathcal{I}} \in \mathcal{E}(\mathcal{I}) \text{ for each } \mathcal{I} \in \mathcal{M}(\Sigma)\}$$

We abbreviate this by writing $\tilde{\mathcal{M}}(\Sigma) = \mathcal{E}(\mathcal{M}(\Sigma))$.

Further the following relation holds between an interpretation $\mathcal{I}$ and $\mathcal{E}(\mathcal{I})$.

Lemma 22. *Let C be a non-epistemic concept, R a non-epistemic role and $\mathcal{I}$ a standard interpretation. Then*

- *for any $x \in \Delta^{\mathcal{I}}$ and for each $\tilde{\mathcal{I}} \in \mathcal{E}(\mathcal{I})$, we have that*

$$x \in C^{\mathcal{I}} \text{ iff } x \in C^{\tilde{\mathcal{I}}}$$

– *for any* $(x, y) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ *and for each* $\tilde{\mathcal{I}} \in \mathcal{E}(\mathcal{I})$*, we have that*

$$(x, y) \in R^{\mathcal{I}} \textit{iff } (x, y) \in R^{\tilde{\mathcal{I}}}$$

Proof Note that as C contains no occurrence of $\mathbf{K}$, the mapping $\varphi_{\tilde{\mathcal{I}}}$ play no role in interpreting C under $\tilde{\mathcal{I}}$. Hence, by simple induction, the proof follows simply from the definition of the extended interpretation. Similarly, for the case of roles, it follows immediately from the definition of the extended interpretations. □

Since the interpretation of any axiom, depends on the interpretation of concept names and role names occuring in it and since the interpretation of a knowledge base, depends on the interpretation of its axioms, as a consequence of the above lemma, we get

Corollary 23. *For any non-epistemic axiom* α *and a standard interpretation* $\mathcal{I}$*, we have that*

$$\mathcal{I} \models \alpha \textit{ iff } \tilde{\mathcal{I}} \Vdash_{\mathrm{e}} \alpha \textit{ for each } \tilde{\mathcal{I}} \in \mathcal{E}(\mathcal{I}).$$

Similarly, for a standard (non-epistemic) knowledge base Σ *and a standard interpretation* $\mathcal{I}$*, we have that*

$$\mathcal{I} \models \Sigma \textit{ iff } \tilde{\mathcal{I}} \Vdash_{\mathrm{e}} \Sigma \textit{ for each } \tilde{\mathcal{I}} \in \mathcal{E}(\mathcal{I}).$$

Now it is easy to see that the (non-epistemic) consequences of a standard knowledge under both, the current semantics and the extended one, coincide.

Corollary 24. *For a given non-epistemic knowledge base* Σ *and a non-epistemic axiom* α*, we have*

$$\Sigma \models \alpha \textit{ if and only if } \Sigma \Vdash_{\mathrm{e}} \alpha$$

As for the common domain assumption and the rigid term assumption, we don't enforce the unique assumption. Hence, this new semantics is more compatible with standard inference engines. Moreover, with this new semantics, the problem, arose when allowing for nominals and universal role in the language of a knowledge base, is avoided. Thus, making it more suitable and appropriate choice for $\mathbf{K}$-extensions of expressive description logics, like $\mathcal{SROIQ}$. In the following section, we extend the translation procedure, presented in Definition 15, in the sense that it allows for $\mathcal{SROIQ}$ as the language of the knowledge base.

6 Deciding Entailment of Extended Epistemic Axioms

In this section, we mainly extend the definition of Φ_Σ (see Definition 15) such that it also handles a richer knowledge base language like, $\mathcal{SROIQ}$. The idea is exactly the same as in case of current semantics presented in Section 3. Nevertheless, there are some slight changes. For the formal proof of the correctness of this procedure, there are several points worth mentioning. Firstly, note that under the new semantics, we don't enforce the UNA, hence, no need to axiomatize it in the knowledge base explicitly. Secondly, we allow for both finite and infinite models of the knowledge base. In fact, a property similar to Lemma 9, can not be proved for $\mathcal{SROIQ}$ knowledge bases i.e., for any finite model of a $\mathcal{SROIQ}$ knowledge base, the existence of an infinite one with similar behaviour is not guaranteed. Hence a complete different approach is taken in proving the correctness of the extended Φ_Σ. More notably, in the current epistemic semantics, Lemma 11 allows us to interchange the role of any two anonymous individuals in a model without compromising its modelhood property. This holds in the extended semantics case as well, but does not suffice in showing formal correctness of Φ_Σ. Instead, we use the definition of $\varphi_{\tilde{\mathcal{I}}}$ for an extended interpretation $\tilde{\mathcal{I}}$.

Like in Section 3, before presenting the extended translation procedure, we first proof two lemmas similar to Lemma 12 and 14. That is, we first show that only named individual can belong to the extension of an epistemic concept of the form $\mathbf{K}C$. Formally,

Lemma 25. *Let Σ be a $\mathcal{SROIQ}$-knowledge base and $C = \mathbf{K}D$ an epistemic concept with $\Sigma \not\models D \equiv \top$. For an extended interpretation $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}(\Sigma)$ and $x \in \Delta^{\tilde{\mathcal{I}}}$, we have that $x \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ iff x is named such that there is an individual name $a \in N_I$ with $x = a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $\Sigma \models D(a)$.*

Proof

" $\Rightarrow$ "

Suppose $x \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. It means that

$$x \in \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})\Big)$$

To contrary suppose that there is no $a \in N_I$ such that $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$ and $\Sigma \not\models D(a)$.

Since $\Sigma \not\models D \equiv \top$, there is a model $\mathcal{I}'$ of Σ such that $\Delta^{\mathcal{I}'} \neq D^{\mathcal{I}'}$. In other words, there is a $y \in \Delta^{\mathcal{I}'}$ with $y \notin D^{\mathcal{I}'}$. By Lemma 22, $y \notin D^{\tilde{\mathcal{I}}'}$

for each $\tilde{\mathcal{I}}' \in \mathcal{E}(\mathcal{I}')$. As by definition, $\mathcal{E}(\mathcal{I}')$ contains all the extended interpretation obtained from $\mathcal{I}'$ by augmenting any mapping from $N_I \cup \mathbb{N}$ to $\Delta^{\mathcal{I}'}$, therefore, there is an extended interpretation $\tilde{\mathcal{J}}' \in \mathcal{E}(\mathcal{I}')$ such that $\varphi_{\tilde{\mathcal{J}}'}^{-1}(y) = \varphi_{\tilde{\mathcal{I}}}^{-1}(x)$ as $\varphi_{\tilde{\mathcal{I}}}$ and $\varphi_{\tilde{\mathcal{J}}'}$ share the same domain, namely $N_I \cup \mathbb{N}$. Since $\mathcal{I}' \models \Sigma$ (as $\mathcal{I}' \in \mathcal{M}(\Sigma)$), by corollary 23 we get that $\tilde{\mathcal{J}}' \Vdash_{\mathrm{e}} \Sigma$ and therefore $\tilde{\mathcal{J}}' \in \tilde{\mathcal{M}}(\Sigma)$. Now $y \notin D^{\tilde{\mathcal{J}}'}$

$$
\begin{array}{lll}
\Rightarrow & \varphi_{\tilde{\mathcal{J}}'}^{-1}(y) \nsubseteq \varphi_{\tilde{\mathcal{J}}'}^{-1}(D^{\tilde{\mathcal{J}}'}) & \\
\Rightarrow & \varphi_{\tilde{\mathcal{J}}'}^{-1}(y) \nsubseteq \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}}) & \text{as } \tilde{\mathcal{J}}' \in \tilde{\mathcal{M}}(\Sigma) \\
\Rightarrow & \varphi_{\tilde{\mathcal{I}}}^{-1}(x) \nsubseteq \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}}) & \text{as } \varphi_{\tilde{\mathcal{I}}}^{-1}(x) = \varphi_{\tilde{\mathcal{J}}'}^{-1}(y) \\
\Rightarrow & x \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})\Big) &
\end{array}
$$

□

which is a contradiction.
" $\Leftarrow$ "
Suppose there is $a \in N_I$ such that $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$ and $\Sigma \models D(a)$. Corollary 23 along with the fact that both Σ and D are non-epistemic implies that $\Sigma \Vdash_{\mathrm{e}} D(a)$. This implies that for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ we have that $a^{\tilde{\mathcal{J}}} \in D^{\tilde{\mathcal{J}}}$, which by definition of $\varphi^{\tilde{\mathcal{J}}}$ implies that $\varphi_{\tilde{\mathcal{J}}}^{-1}(a^{\tilde{\mathcal{J}}}) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})$. Now as $a^{\tilde{\mathcal{J}}} = \varphi_{\tilde{\mathcal{J}}}(a)$ for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we get that $\varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi_{\tilde{\mathcal{J}}}(a)) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})$. This implies that $a \in \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})$ for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. In other words,

$$a \in \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})$$

By definition of $\varphi_{\tilde{\mathcal{I}}}$,

$$\varphi_{\tilde{\mathcal{I}}}(a) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}}))$$

Now as, $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}}$, therefore,

$$a^{\tilde{\mathcal{I}}} \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}}))$$

and hence

$$x \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}}))$$

as $a^{\tilde{\mathcal{I}}} = a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$. By semantics of **K**, we get that $x \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

A similar property can be proved for the roles as well. But again, here we have to be careful about the exceptional case of the roles equivalent to the universal role. The idea is that the extension of a role $\mathsf{K}R$, with R equivalent to the universal role U, and that of the role R, under the extended semantics, coincides. As for the justification, we formulate this as follows.

Claim 26. *Let Σ be a non-epistemic knowledge base. For any non-epistemic role R with $\Sigma \models R \equiv U$ and for each extended interpretation $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}(\Sigma)$, we have:*

$$\mathsf{K}R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$$

Proof Note that by definition

$$\mathsf{K}R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(R^{\tilde{\mathcal{J}}}) \Big) \qquad (*)$$

Since $\Sigma \models R \equiv U$, $R^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for each $\mathcal{I} \in \mathcal{M}(\Sigma)$. Now as both R and Σ are K-free, by Fact 21 we get $\tilde{\mathcal{M}}(\Sigma) = \mathcal{E}(\mathcal{M}(\Sigma))$ and therefore, it follows from Lemma 22 that $R^{\tilde{\mathcal{J}}} = \Delta^{\tilde{\mathcal{J}}} \times \Delta^{\tilde{\mathcal{I}}}$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. This together with $(*)$ implies that

$$\mathsf{K}R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = \varphi_{\tilde{\mathcal{I}}}(N_I \cup \mathbb{N} \times N_I \cup \mathbb{N})$$

as $N_I \cup \mathbb{N}$ is the domain of $\varphi^{\tilde{\mathcal{J}}}$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. Now since $\varphi_{\tilde{\mathcal{I}}}$ is a surjective mapping from $N_I \cup \mathbb{N}$ to $\Delta^{\tilde{\mathcal{I}}}$, we get that $\mathsf{K}R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = \Delta^{\tilde{\mathcal{I}}} \times \Delta^{\tilde{\mathcal{I}}}$ and hence $\mathsf{K}R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = \Delta^{\tilde{\mathcal{I}}} \times \Delta^{\tilde{\mathcal{I}}}$.

Like in case of concepts, we can now show that the extension of every role preceded by K consists only of pairs of individuals with certain characteristics, provided it is not equivalent to the universal role.

Lemma 27. *Let Σ be a $\mathcal{SROIQ}$ knowledge base. Let $R = \mathsf{K}P$ be an epistemic role such that $\Sigma \not\models P \equiv U$. For any extended interpretation $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}(\Sigma)$ and any $x, y \in \Delta^{\tilde{\mathcal{I}}}$, we have that $(x, y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ if and only if at least one of the following holds:*

- *(a) there are individual names $a, b \in N_I$ such that $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$, $b^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = y$ and $\Sigma \models P(a, b)$.*
- *(b) there is an individual name $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ and $\Sigma \models \top \sqsubseteq \exists P^{-}.\{a\}$.*
- *(c) there is an individual name $b \in N_I$ with $b^{\tilde{\mathcal{I}}} = y$ and $\Sigma \models \top \sqsubseteq \exists P.\{b\}$.*

(d) $x = y$ *and* $\Sigma \models \exists P.\mathsf{Self}$.

Proof

" $\Rightarrow$ "

Suppose that $(x, y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ but neither of (a), (b), (c) or (d) hold. We distinguish the following cases:

(1) There are $a, b \in N_I$ with $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$ and $b^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = y$. Regardless of whether $x \neq y$ or $\Sigma \not\models \top \sqsubseteq \exists P.\mathsf{Self}$, since (a) does not hold, we have that $\Sigma \not\models P(a, b)$. It means that there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \Sigma$ but $\mathcal{I} \not\models P(a, b)$ i.e., $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin P^{\mathcal{I}}$. Define an extended interpretation $\tilde{\mathcal{K}}$ as follows:
   - $\Delta^{\tilde{\mathcal{K}}} = \Delta^{\mathcal{I}}$
   - $\cdot^{\tilde{\mathcal{K}}} = \cdot^{\mathcal{I}}$
   - $\varphi_{\tilde{\mathcal{K}}}(c) = c^{\tilde{\mathcal{K}}}$ for each $c \in N_I$

   By definition, $\tilde{\mathcal{K}} \in \mathcal{E}(\mathcal{I})$. As $\mathcal{I} \models \Sigma$, by Corollary 23, we get that $\tilde{\mathcal{I}} \Vdash_{e} \Sigma$ and by Lemma 22, we get that $(a^{\tilde{\mathcal{K}}}, b^{\tilde{\mathcal{K}}}) \notin P^{\tilde{\mathcal{K}}}$. Now by definition of $\varphi^{\tilde{\mathcal{K}}}$ we have that $\varphi^{\tilde{\mathcal{K}}}(c) = c^{\tilde{\mathcal{K}}}$ for each $c \in N_I$ and therefore,

$$\varphi_{\tilde{\mathcal{K}}}^{-1}(\varphi^{\tilde{\mathcal{K}}}(a)) \times \varphi_{\tilde{\mathcal{K}}}^{-1}(\varphi^{\tilde{\mathcal{K}}}(b)) \not\subseteq \varphi_{\tilde{\mathcal{K}}}^{-1}(P^{\tilde{\mathcal{K}}})$$

   This means that

$$(a, b) \notin \varphi_{\tilde{\mathcal{K}}}^{-1}(P^{\tilde{\mathcal{K}}})$$

   Since $\tilde{\mathcal{K}} \in \tilde{\mathcal{M}}(\Sigma)$, we get

$$(a, b) \notin \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}))$$

   By definition of $\varphi_{\tilde{\mathcal{I}}}$,

$$(\varphi_{\tilde{\mathcal{I}}}(a), \varphi_{\tilde{\mathcal{I}}}(b)) \notin \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})))$$

   But $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}} = x$ and $\varphi_{\tilde{\mathcal{I}}}(b) = b^{\tilde{\mathcal{I}}} = y$, therefore

$$(x, y) \notin \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})))$$

   And therefore by semantics of $\mathbf{K}$, we get that $(x, y) \notin \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $\mathbf{K}P = R$, which is a contradiction.

(2) y is anonymous and there is $a \in N_I$ such that $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$. As (d) does not hold, therefore, either $x \neq y$ or $\Sigma \not\models \top \sqsubseteq \exists P.\mathsf{Self}$. In any of the cases, by the assumption it follows from (b) in particular that $\Sigma \not\models \top \sqsubseteq \exists P^-.\{a\}$. Therefore, there is an interpretation $\mathcal{I}$ with $\mathcal{I} \models \Sigma$ but $\mathcal{I} \not\models \top \sqsubseteq \exists P^-.\{a\}$ i.e., there is a $u \in \Delta^{\mathcal{I}}$ such that $u \notin [\exists P^-.\{a\}]^{\mathcal{I}}$ which implies that $(a^{\mathcal{I}}, u) \notin P^{\mathcal{I}}$. By the definition of $\mathcal{E}(\mathcal{I})$, there is an extended interpretation $\tilde{\mathcal{I}}' \in \mathcal{E}(\mathcal{I})$ such that $\varphi_{\tilde{\mathcal{I}}'}^{-1}(u) = \varphi_{\tilde{\mathcal{I}}}^{-1}(y)$. Again this is the case as both $\varphi_{\tilde{\mathcal{I}}}$ and $\varphi_{\tilde{\mathcal{I}}'}$ share the common domain $N_I \cup \mathbb{N}$. Since $(a^{\mathcal{I}}, u) \notin P^{\mathcal{I}}$ and $\tilde{\mathcal{I}}' \in \mathcal{E}(\mathcal{I})$, by Lemma 22, therefore, $(a^{\tilde{\mathcal{I}}'}, u) \notin P^{\tilde{\mathcal{I}}'}$ which implies

$$
\begin{array}{lll}
 & \varphi_{\tilde{\mathcal{I}}'}^{-1}(a^{\tilde{\mathcal{I}}'}) \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \not\subseteq \varphi_{\tilde{\mathcal{I}}'}^{-1}(P^{\tilde{\mathcal{I}}'}) & \\
\Rightarrow & \varphi_{\tilde{\mathcal{I}}'}^{-1}(\varphi^{\tilde{\mathcal{I}}'}(a)) \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \not\subseteq \varphi_{\tilde{\mathcal{I}}'}^{-1}(P^{\tilde{\mathcal{I}}'}) & \text{as } a^{\tilde{\mathcal{I}}'} = \varphi^{\tilde{\mathcal{I}}'}(a) \\
\Rightarrow & \{a\} \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \not\subseteq \varphi_{\tilde{\mathcal{I}}'}^{-1}(P^{\tilde{\mathcal{I}}'}) & \text{as } a^{\tilde{\mathcal{I}}'} = \varphi^{\tilde{\mathcal{I}}'}(a) \\
\Rightarrow & \{a\} \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \not\subseteq \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}) & \text{as } \tilde{\mathcal{I}}' \in \tilde{\mathcal{M}}(\Sigma) \\
\Rightarrow & \{a\} \times \varphi_{\tilde{\mathcal{I}}}^{-1}(y) \not\subseteq \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}) & \text{as } \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) = \varphi_{\tilde{\mathcal{I}}}^{-1}(y) \\
\Rightarrow & (\varphi_{\tilde{\mathcal{I}}}(a), \varphi_{\tilde{\mathcal{I}}}(\varphi_{\tilde{\mathcal{I}}}^{-}(y))) \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big) & \text{by Def. of } \varphi_{\tilde{\mathcal{I}}} \\
\Rightarrow & (\varphi_{\tilde{\mathcal{I}}}(a), y) \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big) & \\
\Rightarrow & (x, y) \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big) & \text{as } \varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}} = x \\
\Rightarrow & (x, y) \notin \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)} = R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} & \text{by semantics}
\end{array}
$$

which is a contradiction.

(3) x is anonymous and there is $b \in N_I$ with $b^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = y$. Again as (d) does not hold, either $x \neq y$ or $\Sigma \not\models \top \sqsubseteq \exists P.\mathsf{Self}$. In either of the cases, it follows from (c) particularly that $\Sigma \not\models \top \sqsubseteq \exists P.\{b\}$. In other words, there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \Sigma$ but $\mathcal{I} \not\models \top \sqsubseteq \exists P.\{b\}$ i.e., there is a $u \in \Delta^{\mathcal{I}}$ such that $u \notin [\exists P.\{b\}]^{\mathcal{I}}$. Consequently, $(u, b^{\mathcal{I}}) \notin P^{\mathcal{I}}$. Again by the definition of $\mathcal{E}(\mathcal{I})$, there is an extended interpretation $\tilde{\mathcal{I}}'$ such that $\varphi_{\tilde{\mathcal{I}}'}^{-1}(u) = \varphi_{\tilde{\mathcal{I}}}^{-1}(x)$. Now by Lemma 22, $(u, b^{\mathcal{I}}) \notin P^{\mathcal{I}}$ implies

$$
\begin{array}{lll}
 & (u, b^{\tilde{\mathcal{I}}'}) \notin P^{\tilde{\mathcal{I}}'} & \\
\Rightarrow & \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(\varphi_{\tilde{\mathcal{I}}'}(b)) \not\subseteq \varphi_{\tilde{\mathcal{I}}'}^{-1}(P^{\tilde{\mathcal{I}}'}) & \\
\Rightarrow & \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(\varphi_{\tilde{\mathcal{I}}'}(b)) \not\subseteq \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}) & \text{as } \tilde{\mathcal{I}}' \in \tilde{\mathcal{M}}(\Sigma) \\
\Rightarrow & \varphi_{\tilde{\mathcal{I}}}^{-1}(x) \times \{b\} \notin \bigcap_{\tilde{\mathcal{J}} \subseteq \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}) & \text{as } \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) = \varphi_{\tilde{\mathcal{I}}}^{-1}(x) \\
\Rightarrow & (\varphi_{\tilde{\mathcal{I}}}(\varphi_{\tilde{\mathcal{I}}}^{-1}(x)), \varphi_{\tilde{\mathcal{I}}}(b)) \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big) & \\
\Rightarrow & (x, y) \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big) & \text{as } \varphi_{\tilde{\mathcal{I}}}(b) = b^{\tilde{\mathcal{I}}} = y \\
\Rightarrow & (x, y) \notin \mathbf{K}P^{\mathcal{I},\tilde{\mathcal{M}}(\Sigma)} = R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} & \text{by semantics}
\end{array}
$$

$\square$

which is a contradiction.

" $\Leftarrow$ "

Suppose either (a), (b), (c) or (d) holds. We have to show then $(x,y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. We make the following case distinction:

(1) There are $a,b \in N_I$ such that $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$ and $b^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = y$ and $\Sigma \models P(a,b)$:
Since both P and Σ contain no occurrence of $\mathbf{K}$, by Corollary 24 we get that $\Sigma \Vdash_{\mathrm{e}} P(a,b)$ i.e., for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, $(a^{\tilde{\mathcal{J}}}, b^{\tilde{\mathcal{J}}}) \in P^{\tilde{\mathcal{J}}}$ which implies that

$$\varphi_{\tilde{\mathcal{J}}}^{-1}(a^{\tilde{\mathcal{J}}}) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(b^{\tilde{\mathcal{J}}}) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$$

for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. As $c^{\tilde{\mathcal{J}}} = \varphi_{\tilde{\mathcal{J}}}(c)$ for $c \in N_I$ and $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, therefore,

$$\varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi_{\tilde{\mathcal{J}}}(a)) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi_{\tilde{\mathcal{J}}}(b))) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$$

for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ i.e.,

$$(a,b) \in \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}))$$

which, using the definition of $\varphi_{\tilde{\mathcal{I}}}$, implies that

$$(\varphi_{\tilde{\mathcal{I}}}(a), \varphi_{\tilde{\mathcal{I}}}(b)) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})))$$

Finally, as $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}} = x$ and $\varphi_{\tilde{\mathcal{I}}}(b) = b^{\tilde{\mathcal{I}}} = y$, therefore,

$$(x,y) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}))) = \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$$

Hence, $(x,y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $R = \mathbf{K}P$.

(2) There is an individual $a \in N_I$ with $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$ and $\Sigma \models \top \sqsubseteq \exists P^{-}.\{a\}$:
By Corollary 24 and the fact that $\Sigma \models \top \sqsubseteq \exists P^{-}.\{a\}$, we get that $\Sigma \Vdash_{\mathrm{e}} \top \sqsubseteq \exists P^{-}.\{a\}$ as neither Σ nor $\top \sqsubseteq \exists P^{-}.\{a\}$ contains any occurrence of $\mathbf{K}$. Hence, for each extended interpretation $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we get that $\tilde{\mathcal{J}} \models \top \sqsubseteq \exists P^{-}.\{a\}$, i.e., every $u \in \Delta^{\tilde{\mathcal{J}}}$ is such that $u \in [\exists P^{-}.\{a\}]^{\tilde{\mathcal{J}}}$. This means that for every $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $u \in \Delta^{\tilde{\mathcal{J}}}$,

we have that $(a^{\tilde{\mathcal{J}}}, u) \in P^{\tilde{\mathcal{J}}}$. Now, using the definition of $\varphi_{\tilde{\mathcal{J}}}^{-1}$, for $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we get that

$$\varphi_{\tilde{\mathcal{J}}}^{-1}(a^{\tilde{\mathcal{J}}}) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(u) = \varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi^{\tilde{\mathcal{J}}}(a)) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(u) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$$

for any $u \in \Delta^{\tilde{\mathcal{I}}}$ and $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. Note that $a \in \varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi_{\tilde{\mathcal{J}}}(a))$. Further since $\varphi_{\tilde{\mathcal{J}}}$ has domain $N_I \cup \mathbb{N}$ and $u \in \Delta^{\tilde{\mathcal{J}}}$ is arbitrary, for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we get $(a, t) \in \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$ for each $t \in (N_I \cup \mathbb{N})$ i.e.,

$$(a, t) \in \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$$

Using the definition of $\varphi_{\tilde{\mathcal{I}}}$, therefore,

$$(\varphi_{\tilde{\mathcal{I}}}(a), \varphi_{\tilde{\mathcal{I}}}^{-1}(t)) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}))$$

Since $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}} = x$ and $\varphi_{\tilde{\mathcal{I}}}$ is a surjective mapping with range $\Delta^{\tilde{\mathcal{I}}}$, therefore, for every $v \in \Delta^{\tilde{\mathcal{I}}}$ we get that

$$(x, v) \in \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big)$$

In particular,

$$(x, y) \in \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big)$$

which, by semantics implies that $(x, y) \in \mathbf{K}P^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as $R = \mathbf{K}P$.

(3) There is $b \in N_I$ with $b^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = y$ and $\Sigma \models \top \sqsubseteq \exists P.\{b\}$:
By Corollary 24, we get aht $\Sigma \Vdash_{e} \top \sqsubseteq \exists P.\{b\}$ as both Σ and $\top \sqsubseteq \exists P.\{b\}$ are $\mathbf{K}$-free. In other words, for every $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we have that $(u, b^{\tilde{\mathcal{J}}}) \in P^{\tilde{\mathcal{J}}, \tilde{\mathcal{M}}(\Sigma)}$ for any $u \in \Delta^{\tilde{\mathcal{J}}}$. Now as $\varphi^{\tilde{\mathcal{J}}}(b) = b^{\tilde{\mathcal{J}}}$, using the definition of $\varphi_{\tilde{\mathcal{J}}}^{-1}$ we get that

$$\varphi_{\tilde{\mathcal{J}}}^{-1}(u) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi^{\tilde{\mathcal{J}}}(b)) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$$

for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $u \in \Delta^{\tilde{\mathcal{J}}}$. Again as $b \in \varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi_{\tilde{\mathcal{J}}}(b))$ and $\varphi_{\tilde{\mathcal{J}}}$ has domain $N_I \cup \mathbb{N}$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we get $(t, b) \in \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$ and therefore,

$$(t, b) \in \varphi_{\tilde{\mathcal{J}}}^{-1} \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (P^{\tilde{\mathcal{J}}})$$

for each $t \in (N_I \cup \mathbb{N})$. Using definition of $\varphi_{\tilde{\mathcal{I}}}$ we get

$$(\varphi_{\tilde{\mathcal{I}}}(t), \varphi_{\tilde{\mathcal{I}}}(b)) \in \varphi_{\tilde{\mathcal{I}}}\big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\big)$$

for any $t \in (N_I \cup \mathbb{N})$. Since $\varphi_{\tilde{\mathcal{I}}}(b) = b^{\tilde{\mathcal{J}}} = y$ and $\varphi_{\tilde{\mathcal{I}}}$ is a surjective mapping with range $\Delta^{\tilde{\mathcal{I}}}$, we get that

$$(v, y) \in \varphi_{\tilde{\mathcal{I}}}\big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\big)$$

for any $v \in \Delta^{\tilde{\mathcal{I}}}$. In particular,

$$(x, y) \in \varphi_{\tilde{\mathcal{I}}}\big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\big)$$

which by semantics implies that $(x, y) \in R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as $R = \mathbf{K}P$.

(4) $x = y$ and $\Sigma \models \top \sqsubseteq \exists P.\mathsf{Self}$:
As both Σ and the axiom $\top \sqsubseteq \exists P.\mathsf{Self}$ are $\mathbf{K}$-free, Corollary 24 implies that $\Sigma \Vdash_{e} \top \sqsubseteq \exists P.\mathsf{Self}$ i.e., for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $u \in \Delta^{\tilde{\mathcal{J}}}$, we have that $(u, u) \in P^{\tilde{\mathcal{J}}}$ and by definition of $\varphi_{\tilde{\mathcal{J}}}^{-1}$, therefore, $\varphi_{\tilde{\mathcal{J}}}^{-1}(u) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(u) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$. But as $\varphi_{\tilde{\mathcal{J}}}$, for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, is a mapping with domain $N_I \cup \mathbb{N}$, we get $(t, t) \in \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $t \in (N_I \cup \mathbb{N})$. In other words,

$$(t, t) \in \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}, \tilde{\mathcal{M}}(\Sigma)})$$

for each $t \in (N_I \cup \mathbb{N})$. Now using the definition of $\varphi_{\tilde{\mathcal{I}}}$, we get that

$$(\varphi_{\tilde{\mathcal{I}}}(t), \varphi_{\tilde{\mathcal{I}}}(t)) \in \varphi_{\tilde{\mathcal{I}}}\big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} P^{\tilde{\mathcal{J}}, \tilde{\mathcal{M}}(\Sigma)}\big) = \mathbf{K}P^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$$

for each $t \in (N_I \cup \mathbb{N})$. As $\varphi_{\tilde{\mathcal{I}}}$ is a surjective mapping from $N_I \cup \mathbb{N}$ to $\Delta^{\tilde{\mathcal{I}}}$, therefore, $(v, v) \in \mathbf{K}P^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ for every $v \in \Delta^{\tilde{\mathcal{I}}}$. In particular, $(x, x) \in \mathbf{K}P^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ and therefore $(x, x) \in R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as $R = \mathbf{K}P$.

The correspondence we establish in Lemma 25 and 27 lets us to extend the translation procedure Φ_Σ of Definition 15 in a way that it maps (complex) epistemic concept expressions to non-epistemic ones which are equivalent in all models of the given $\mathcal{SROIQ}$ knowledge base Σ. We represent this extension by $\tilde{\Phi}_\Sigma$.s

Definition 28. Given a $\mathcal{SROIQ}$ knowledge base Σ, we define a function $\tilde{\Phi}_\Sigma$ mapping $\mathcal{SROIQK}$ concept expressions to $\mathcal{SROIQ}$ concept expressions (where we let $\{\} = \emptyset = \bot$):

$$
\tilde{\Phi}_\Sigma : \begin{cases}
C \mapsto C \quad \text{if } C \text{ is an atomic or one-of concept, } \top \text{ or } \bot; \\
\mathbf{K}D \mapsto \begin{cases} \top & \text{if } \Sigma \models \tilde{\Phi}_\Sigma(D) \equiv \top \\ \{a \in N_I \mid \Sigma \models \tilde{\Phi}_\Sigma(D)(a)\} & \text{otherwise} \end{cases} \\
\exists \mathbf{K}S.\mathsf{Self} \mapsto \begin{cases} \exists S.\mathsf{Self} & \text{if } \Sigma \models \top \sqsubseteq \exists S.\mathsf{Self} \\ \{a \in N_I \mid \Sigma \models S(a,a)\} & \text{otherwise} \end{cases} \\
C_1 \sqcap C_2 \mapsto \tilde{\Phi}_\Sigma(C_1) \sqcap \tilde{\Phi}_\Sigma(C_2) \\
C_1 \sqcup C_2 \mapsto \tilde{\Phi}_\Sigma(C_1) \sqcup \tilde{\Phi}_\Sigma(C_2) \\
\neg C \mapsto \neg \tilde{\Phi}_\Sigma(C) \\
\exists R.D \mapsto \exists R.\tilde{\Phi}_\Sigma(D) \quad \text{for non-epistemic role } R \\
\exists \mathbf{K}P.D \mapsto \begin{cases} \bigsqcup_{a \in N_I} \{a\} \sqcap \exists P.(\{b \in N_I \mid \Sigma \models P(a,b) \sqcap \tilde{\Phi}_\Sigma(D)\}) \\ \sqcup \exists P.\big(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D)\big) \\ \sqcup \{a \in N_I \mid \Sigma \models \top \sqsubseteq \exists P^-\{a\}\} \sqcap \exists P.\tilde{\Phi}_\Sigma(D) \\ \sqcup \begin{cases} \tilde{\Phi}_\Sigma(D) & \text{if } \Sigma \models \top \sqsubseteq \exists P.\mathsf{Self} \\ \bot & \text{otherwise} \end{cases} \end{cases} \\
\forall R.D \mapsto \forall R.\tilde{\Phi}_\Sigma(D) \quad \text{for non-epistemic role } R; \\
\forall \mathbf{K}P.D \mapsto \neg \tilde{\Phi}_\Sigma(\exists \mathbf{K}P.\neg D) \\
\geqslant nS.D \mapsto \geqslant nS.\tilde{\Phi}_\Sigma(D) \quad \text{for non-epistemic role } S; \\
\geqslant n\mathbf{K}S.D \mapsto \begin{cases} \bigsqcup_{a \in N_I} \{a\} \sqcap \geqslant nS.(\{b \in N_I \mid \Sigma \models S(a,b) \sqcap \tilde{\Phi}_\Sigma(D)\}) \\ \sqcup \{a \in N_I \mid \Sigma \models \top \sqsubseteq \exists S^-.\{a\}\} \sqcap \geqslant nS.\tilde{\Phi}_\Sigma(D) \\ \sqcup \geqslant nS.(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D)) \\ \sqcup \begin{cases} \geqslant (n-1)S.\big(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D)\big) \sqcap \\ \tilde{\Phi}_\Sigma(D) \sqcap \neg \{a \mid a \in N_I\} & \text{if } \Sigma \models \top \sqsubseteq \exists S.\mathsf{Self} \\ \bot & \text{otherwise} \end{cases} \end{cases} \\
\leqslant nS.D \mapsto \leqslant nS.\tilde{\Phi}_\Sigma(D) \quad \text{for non-epistemic role } S; \\
\leqslant n\mathbf{K}S.D \mapsto \neg \tilde{\Phi}_\Sigma(\geqslant (n{+}1)\mathbf{K}S.D) \\
\Xi \mathbf{K}R.D \mapsto \Xi R.\tilde{\Phi}_\Sigma(D) \quad \text{for } \Xi \in \{\forall, \exists, \geqslant n, \leqslant n\} \text{ and } \Sigma \models R \equiv U
\end{cases}
$$

◊

We now prove that a method based on the translation function $\tilde{\Phi}_\Sigma$ as in Definition 28 is indeed correct. In the following Lemma, we show that the extension of a $\mathcal{SROIQK}$ concept and the extension of $\mathcal{SROIQ}$ concept obtained using the translation function $\tilde{\Phi}_\Sigma$, agree under each extended interpretation in $\tilde{\mathcal{M}}(\Sigma)$.

Lemma 29. *Let Σ be a $\mathcal{SROIQ}$-knowledge base and C be a $\mathcal{SROIQK}$ concept. Then for any extended interpretation $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}(\Sigma)$, we have that $C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = [\tilde{\Phi}_\Sigma(C)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.*

Proof For the proof we use induction on the structure of C and show that for each $x \in \Delta^{\tilde{\mathcal{I}}}$, we have that $x \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ if and only if $x \in$

$(\tilde{\Phi}_\Sigma(C))^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. For the base case; C is atomic or one-of concept, and the cases where $C = \top$ and $C = \bot$, the lemma follows immediately from the definition of $\tilde{\Phi}_\Sigma$. For the cases, where $C = C_1 \sqcap C_2$, $C = C_1 \sqcup C_2$ or $C = \neg D$, it follows from the induction hypothesis. The non-trivial cases are considered in the following.

(i) $C = \mathbf{K}D$ and $\Sigma \not\models D \equiv \top$:
By Lemma 27, $x \in \mathbf{K}D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ iff there is an $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ and $\Sigma \models D(a)$. This is equivalent to $x \in \{a \in N_I \mid \Sigma \models D(a)\}^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and hence, by definition of $\tilde{\Phi}_\Sigma$, to $x \in [\tilde{\Phi}_\Sigma(\mathbf{K}D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(ii) $C = \mathbf{K}D$ and $\Sigma \models D \equiv \top$:
By Corollary 24 we get hat $\Sigma \Vdash_{\mathrm{e}} D \equiv \top$ i.e., for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ we have that $D^{\tilde{\mathcal{J}},\tilde{\mathcal{M}}(\Sigma)} = \Delta^{\tilde{\mathcal{I}}}$. Consequently, we get

$$\varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}},\tilde{\mathcal{M}}(\Sigma)})\Big) = \varphi_{\tilde{\mathcal{I}}}(N_I \cup \mathbb{N})$$

as $\varphi_{\tilde{\mathcal{I}}}^{-1}$ is a surjective mapping from $N_I \cup \mathbb{N}$ to $\Delta^{\tilde{\mathcal{J}}}$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. Now $\varphi_{\tilde{\mathcal{I}}}(N_I \cup \mathbb{N})$ yields $\Delta^{\tilde{\mathcal{I}}}$ as it is a surjective mapping to $\Delta^{\tilde{\mathcal{I}}}$. Hence we get that $\mathbf{K}D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = \Delta^{\tilde{\mathcal{I}}} = \top^{\tilde{\mathcal{I}}}$, which by definition of $\tilde{\Phi}_\Sigma$ yields that $\mathbf{K}D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = [\tilde{\Phi}_\Sigma(\mathbf{K}D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Consequently, $x \in \mathbf{K}D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ iff $x \in \tilde{\Phi}_\Sigma(\mathbf{K}D)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(iii) $C = \exists \mathbf{K}S.\mathsf{Self}$:
" $\Rightarrow$ "
Suppose that $\Sigma \models \top \sqsubseteq \exists S.\mathsf{Self}$. By semantics, $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that for each $\tilde{\mathcal{J}} \models \tilde{\mathcal{M}}(\Sigma)$ we have that $(x,x) \in S^{\tilde{\mathcal{J}},\tilde{\mathcal{M}}(\Sigma)}$. In particular, $(x,x) \in S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ i.e., $x \in [\exists S.\mathsf{Self}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Therefore, $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as by definition $\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \exists S.\mathsf{Self}$.
Suppose that it is not the case that $\Sigma \models \top \sqsubseteq \exists S.\mathsf{Self}$. By semantics, $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies $(x,x) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. By Lemma 27, there is $a^{\tilde{\mathcal{I}}} = x$ such that $\Sigma \models S(a,a)$. Hence, $a \in \{c \in N_I \mid \Sigma \models S(c,c)\}$, which implies that $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as by definition $\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \{c \in N_I \mid \Sigma \models S(c,c)\}$.
" $\Leftarrow$ "
Suppose that $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Based on the definition of $\tilde{\Phi}_\Sigma$ we distinguish the following cases.
- $\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \exists S.\mathsf{Self}$:
  Like in (4) of Lemma 27, we can show that $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

– $\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \{c \in N_I \mid \Sigma \models S(c,c)\}$:
$x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that there some $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ such that $\Sigma \models S(a,a)$. By Lemma 27, it means that $(a^{\tilde{\mathcal{I}}}, a^{\tilde{\mathcal{I}}}) = (x,x) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and hence $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(iv) $C = \exists P.D$ and P is a non-epistemic role:
By the semantics, $x \in [\exists P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ iff there is a $y \in \Delta^{\tilde{\mathcal{I}}}$ with $(x,y) \in P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $y \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, therefore by induction, $y \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Hence, it is equivalent to $x \in [\exists P.\tilde{\Phi}\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which, by definition, is the case if and only if $x \in [\tilde{\Phi}_\Sigma(\exists P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(v) $C = \exists \mathbf{K}P.D$:
<u>$x \in (\exists \mathbf{K}P.D)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}P.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$:</u>
By semantics, $x \in (\exists \mathbf{K}P.D)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ means that there is a $y \in \Delta^{\tilde{\mathcal{I}}}$ with $(x,y) \in (\mathbf{K}P)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $y \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ hence, by induction $y \in \tilde{\Phi}\Sigma(D)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. By Lemma 27, $(x,y) \in \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}}$ implies at least one of the following should hold:

– there are $a,b \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$, $b^{\tilde{\mathcal{I}}} = x$ and $\Sigma \models P(a,b)$. This means that $b \in \{c \in N_I \mid \Sigma \models P(a,c)\}$, therefore,

$$b^{\tilde{\mathcal{I}}} \in [\{c \in N_I \mid \Sigma \models P(a,c)\}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$$

Now as $y \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, therefore,

$$x \in [\{a\} \sqcap \exists P.(\{c \in N_I \mid \Sigma \models P(a,c)\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$$

which by definition of $\tilde{\Phi}_\Sigma$ implies that $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}P.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$

– there is an $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ and $\Sigma \models \top \sqsubset \exists P^-.\{a\}$. This means that $a \in \{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P^-\{c\}\}$, therefore,

$$a^{\tilde{\mathcal{I}}} \in [\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P^-\{c\}\}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$$

Now as Σ is $\mathbf{K}$-free, by Corollary 24 $\Sigma \models \top \sqsubseteq \exists P^-.\{a\}$ implies that $\Sigma \models_{e} \top \sqsubseteq \exists P^-.\{a\}$ i.e., $(a^{\tilde{\mathcal{J}}}, u) \in P^{\tilde{\mathcal{J}},\tilde{\mathcal{M}}(\Sigma)}$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $u \in \Delta^{\tilde{\mathcal{J}}}$. In particular, $(a^{\tilde{\mathcal{I}}}, y) = (x,y) \in P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Consequently, we get that $x \in [\exists P.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $y \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. This along with $(*)$ implies that $x = a^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}P.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as per definition of $\tilde{\Phi}_\Sigma$.

– there is a $b \in N_I$ with $b^{\tilde{\mathcal{I}}} = y$ and $\Sigma \models \top \sqsubseteq \exists P.\{b\}$ i.e., $b \in \{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{c\}\}$ which implies that

$$b^{\tilde{\mathcal{I}}} \in [\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{c\}\}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \qquad (*)$$

Since Σ is $\mathbf{K}$-free we get from Corollary 24 that $\Sigma \models_{\mathrm{e}} \top \sqsubseteq \exists P.\{b\}$ i.e., for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $u \in \Delta^{\tilde{\mathcal{I}}}$ we have that $(u, b^{\tilde{\mathcal{J}}}) \in P^{\tilde{\mathcal{J}},\tilde{\mathcal{M}}(\Sigma)}$. In particular, $(x, b^{\tilde{\mathcal{I}}}) = (x, y) \in P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, which along with $(*)$ and the fact that $y \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ immediately implies that $x \in [\exists \mathbf{K}P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as per definition of $\tilde{\Phi}_\Sigma$.

– $x = y$ and $\Sigma \models \top \sqsubseteq \exists P.\mathsf{Self}$. As $\tilde{\Phi}_\Sigma(\exists \mathbf{K}P.D) = \tilde{\Phi}_\Sigma(D)$, therefore, we get that $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}P.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $x = y \in [\tilde{\Phi}_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

<u>$x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies $x \in [\exists \mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$:</u>

According to the definition of $\tilde{\Phi}_\Sigma$, we make the following case distinction.

– there is an $a \in N_I$ such that $a^{\tilde{\mathcal{I}}} = x$ and $x \in [\exists P.(\{c \in N_I \mid \Sigma \models P(a, c)\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ i.e., there is some $b \in N_I$ such that $b^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $\Sigma \models P(a, b)$. This, by Lemma 27, implies that $(a^{\tilde{\mathcal{I}}}, b^{\tilde{\mathcal{I}}}) \in \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Hence, we get that $x = a^{\tilde{\mathcal{I}}} \in [\exists \mathbf{K}P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $b^{\tilde{\mathcal{I}}} \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ by induction.
– $x \in [\exists P.(\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{c\}\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which implies that there is some $b \in N_I$ such that $\Sigma \models \top \sqsubseteq \exists P.\{b\}$ and $b^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. It follows from Lemma 27, that $(x, b^{\tilde{\mathcal{I}}}) \in \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which immediately implies that $x \in [\mathbf{K}P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $b^{\tilde{\mathcal{I}}} \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ by induction.
– there is an $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ such that $\Sigma \models \top \sqsubseteq \exists P^-.\{a\}$ and $x \in [\exists P.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. It means that there is a $y \in \Delta^{\tilde{\mathcal{I}}}$ such that $y \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. By Lemma 27, we get that $(x, y) \in \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and hence, by semantics, $x \in [\exists \mathbf{K}P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $y \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ by induction.
– $x \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, therefore by induction $x \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Since it is the case that $\Sigma \models \exists P.\mathsf{Self}$, Lemma 27 implies that $(x, x) \in \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and therefore, $x \in [\exists \mathbf{K}P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(vi) $C = \geqslant n\mathbf{K}S.D$

<u>$x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies $x \in [\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$</u>

By semantics, $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that there are pair-wise distinct $y_1, \dots, y_m \in \Delta^{\tilde{\mathcal{I}}}$ with $m \geq n$ such that $(x, y_i) \in \mathbf{K}S^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ and $y_i \in D^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$, therefore by induction, $y_i \in \tilde{\Phi}_\Sigma(D)^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. By Lemma 27, this implies that at least one of the following should hold:

- there are $a, b_1, \dots, b_m$ with $a^{\tilde{\mathcal{I}}} = x$ and $b_i^{\tilde{\mathcal{I}}} = y_i$ such that $\Sigma \models S(a, b_i)$ for $i \leq m$ i.e., $b_i \in \{c \in N_I \mid \Sigma \models (a, c)\}$. Since Σ is $\mathbf{K}$-free, it follows from Corollary 24 that $\Sigma \Vdash_{\mathrm{e}} S(a, b_i)$ for $i \leq m$. This implies that $(a^{\tilde{\mathcal{I}}}, b_i^{\tilde{\mathcal{I}}}) = (x, y_i) \in S^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ for each $i \leq m$. As $m \geq n$ and $b_i \in \{c \in N_I \mid \Sigma \models S(a, c)\}$ with $y_i = b_i^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$, therefore, we get that $a^{\tilde{\mathcal{I}}} \in [\geqslant nS.(\{c \in N_I \mid \Sigma \models S(a, c)\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ which immediately implies that $x \in [\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as $x \in \{a\}^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$.
- there is an $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ such that $\Sigma \models \top \sqsubseteq \exists S^-.\{a\}$. This implies that

  $$a \in \{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S^-.\{c\}\} \qquad (*)$$

  Now by Corollary 24, $\Sigma \models \top \sqsubseteq \exists S^-.\{a\}$ implies that $\Sigma \Vdash_{\mathrm{e}} \top \sqsubseteq \exists S^-.\{a\}$. It means that for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we have that $(a^{\tilde{\mathcal{J}}}, v) \in S^{\tilde{\mathcal{J}}}$ for arbitrary $v \in \Delta^{\tilde{\mathcal{J}}}$. In particular, $(x, y_i) \in S^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as $a^{\tilde{\mathcal{I}}} = x$ and $y_i \in \Delta^{\tilde{\mathcal{I}}}$ for $i \leq m$. Now since $y_i \in \tilde{\Phi}_\Sigma(D)^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ and $m \geq n$, it follows from the semantics that $x \in [\geqslant nS.\Phi_\Sigma(D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$. This along with $(*)$ implies that $x \in [\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S^-.\{c\} \sqcap \geqslant nS.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ and therefore, by definition of $\tilde{\Phi}_\Sigma$, we get that $x \in [\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D)]$.
- there are $b_1, \dots, b_m \in N_I$ with $b_i^{\tilde{\mathcal{I}}} = y_i$ and $\Sigma \models \top \sqsubseteq \exists S.\{b_i\}$ for $i \leq m$. This means that

  $$\{b_1, \dots, b_m\} \subseteq \{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{c\}\} \qquad (*)$$

  Now by Corollary 24, $\Sigma \models \top \sqsubseteq \exists S.\{b_i\}$ implies that $\Sigma \Vdash_{\mathrm{e}} \top \sqsubseteq \exists S.\{b_i\}$ for $i \leq m$. Therefore, we get that for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, $(u, b_i^{\tilde{\mathcal{J}}}) \in S^{\tilde{\mathcal{J}}, \tilde{\mathcal{M}}(\Sigma)}$ for arbitrary $u \in \Delta^{\tilde{\mathcal{I}}}$ and $i \leq m$. In particular, we have that for each $i \leq m$, $(x, b_i^{\tilde{\mathcal{I}}}) \in S^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$. Now as $b_i^{\tilde{\mathcal{I}}} = y_i \in \tilde{\Phi}_\Sigma(D)^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$, consequently, it follows from $(*)$, that

  $$x \in [\geqslant nS.(\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{c\}\} \sqcap \Phi_\Sigma(D))]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$$

  as $m \geq n$ and therefore, $x \in [\Phi_\Sigma(\geqslant n\mathbf{K}S.D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as per definition of $\tilde{\Phi}_\Sigma$.

- there is a $y \in \{y_1, \ldots, y_m\}$ such that $x = y$ and $\Sigma \models \top \sqsubseteq \exists S.\mathsf{Self}$. Hence we have that

$$x \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \qquad (*)$$

  Note that if x is named, we can proceed as the first two cases of the proof. Here hence, we assume that x is unnamed i.e.,

$$x \in [\neg\{a \mid a \in N_I\}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \qquad (**)$$

  Suppose that $x \notin [\geqslant(n-1)S.(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. This means that there are $b_1, \ldots, b_k \in N_I$ with $k < (n-1)$such that $b_i \in [\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Note that $\Sigma \models \top \sqsubseteq \exists S.\mathsf{Self}$, by Corollary 24 implies that $\Sigma \Vdash_{\mathrm{e}} \top \sqsubseteq \exists S.\mathsf{Self}$. Hence, we get that $(x,x) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which along with the fact that $x = y \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that $x[\geqslant 1S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. By the assumption here we get that there are distinct $z_1, \ldots, z_{m'} \in \Delta^{\tilde{\mathcal{I}}}$ with $(x, z_i) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $z_i \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m'$ and m' is at most $(n-1)$. Which is a contradiction as $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Therefore, it must be the case that $x \in [\geqslant(n-1)S.(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which along with $(*)$ and $(**)$ implies that $x \in [\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as per definition of $\tilde{\Phi}_\Sigma$.

<u>$x \in [\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$</u>

According to the definition of Φ_Σ, at least one of the following is the case.

- there are $a, b_1, \ldots, b_m \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ such that $\Sigma \models S(a, b_i)$ and $b_i^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. By Lemma 27, therefore, $(x, b_i^{\tilde{\mathcal{I}}}) = (a^{\tilde{\mathcal{I}}}, b_i^{\tilde{\mathcal{I}}}) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and by induction $b_i^{\tilde{\mathcal{I}}} \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. This immediately implies that $x = a^{\tilde{\mathcal{I}}} \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $m \geq n$.
- $\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D) = \{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S^-.\{c\}\} \sqcap \geqslant nS.\tilde{\Phi}_\Sigma(D)$:
  $x \in [\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S^-.\{c\}\} \sqcap \geqslant nS.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that there is an $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ and $a^{\tilde{\mathcal{I}}} \in [\geqslant nS.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ i.e., there are pair-wise disjoint $y_1, \ldots, y_m \in \Delta^{\tilde{\mathcal{I}}}$ with $m \geq n$ such that $y_i \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. By induction, therefore, for $i \leq m$ we have that $y_i \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Now $\Sigma \models \top \sqsubseteq \exists S^-.\{a\}$ implies that $(x,y) \in S^{\tilde{\mathcal{I}}}$ as $a^{\tilde{\mathcal{I}}} = x$ and therefore, by Lemma 27, $(x,y) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for each $y \in \Delta^{\tilde{\mathcal{I}}}$. In particular,

$(x, y_i) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. As $y_i \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$ and $m \geq n$, consequently we get that $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

- $\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D) = \geqslant nS.(\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{c\}\} \sqcap \tilde{\Phi}_\Sigma(D))$: $x \in [\geqslant nS.(\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S\cdot\{c\}\} \sqcap \Phi_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that there are distinct $b_1, \ldots, b_m \in N_I$ with $m \geq n$, such that $\Sigma \models \top \sqsubseteq \exists S.\{b_i\}$ and $b_i^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, therefore by induction $b_i^{\tilde{\mathcal{I}}} \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. By Lemma 27, $\Sigma \models \top \sqsubseteq \exists S.\{b_i\}$ implies that $(x, b_i^{\tilde{\mathcal{I}}}) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$ which immediately yields that $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.
- x is anonymous with $x \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $x \in [\geqslant(n-1)S.(\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{c\}\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, which, as already proved in the previous case, implies

$$x \in [\geqslant(n-1)\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \qquad (*)$$

Now by Lemma 27, we have that $(x, x) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. This along with $(*)$ implies that $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $x \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ by induction.

(vii) $C = \Xi\mathbf{K}R.D$ for $\Xi \in \{\forall, \exists, \geqslant n, \leqslant n\}$ and $\Sigma \models R \equiv U$:
By Claim 26, we have that $\mathbf{K}R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Hence, by induction, it follows immediately that $x \in [\Xi\mathbf{K}R.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ if and only if $x \in [\Xi R.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = [\tilde{\Phi}_\Sigma(\Xi\mathbf{K}R.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(viii) the rest of the cases can be proved analogously. □

In the following, based on Lemma 29 we can show the result that the translation function $\tilde{\Phi}_\Sigma$ indeed can be used to reduce the problem of entailment in $\mathcal{SROIQK}$ axioms by $\mathcal{SROIQ}$ knowledge bases to the problem of entailment of $\mathcal{SROIQ}$ axioms. Formally,

Theorem 30. *For a $\mathcal{SROIQ}$ knowledge base Σ, $\mathcal{SROIQK}$ concept C, D, and an individual a, the following hold:*

1 $\Sigma \mathrel{|\!\!\models_{\mathrm{e}}} C(a)$ if and only if $\Sigma \models \tilde{\Phi}_\Sigma(C)(a)$.
2 $\Sigma \mathrel{|\!\!\models_{\mathrm{e}}} C \sqsubseteq D$ if and only if $\Sigma \models \tilde{\Phi}_\Sigma(C) \sqsubseteq \tilde{\Phi}_\Sigma(D)$.

Proof. For the first case, note that $\Sigma \mathrel{|\!\!\models_{\mathrm{e}}} C(a)$ is equivalent to $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which by Lemma 29 implies that $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \in \tilde{\Phi}_\Sigma(C)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for all $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}(\Sigma)$. Since Σ and $\tilde{\Phi}_\Sigma(C)$ are $\mathbf{K}$-free, by Fact 21 we get $\tilde{\mathcal{M}} = \mathcal{E}(\mathcal{M}(\Sigma))$ and therefore by Lemma 22 we get the equivalent statement that $a^{\mathcal{I}} \in \tilde{\Phi}_\Sigma(C)^{\mathcal{I}}$ for each $\mathcal{I} \in \mathcal{M}(\Sigma)$ and therefore, $\Sigma \models C(a)$. We can prove the second case with similar arguments. □

Note that the definition of Φ_Σ and $\tilde{\Phi}_\Sigma$ coincide for a $\mathcal{SRIQ}$ knowledge base Σ. This allows us to establish a correspondence between both notions of entailment, i.e., entailment under the current semantics ($\Vdash$) and entailment under the extended semantics ($\Vdash_{\mathrm{e}}$). Formally,

Corollary 31. *For a given $\mathcal{SRIQ}$ knowledge base Σ, epistemic concepts C, D and an individual name a, we have that*

1. *$\Sigma \Vdash_{\mathrm{e}} C(a)$ under the unique name assumption if and only if $\Sigma \Vdash C(a)$, similarly*
2. *$\Sigma \Vdash_{\mathrm{e}} C \sqsubseteq D$ under the unique name assumption if and only if $\Sigma \Vdash C \sqsubseteq D$*

Proof. For the proof of the first part, note that by Theorem 17 $\Sigma \Vdash C(a)$ is equivalent to $\Sigma \Vdash \Phi_\Sigma(C)(a)$. Since Σ is a $\mathcal{SRIQ}$ knowledge base and that Φ_Σ behaves similar to $\tilde{\Phi}_\Sigma$ (as the unique name assumption is satisfied), therefore we get that $\Sigma \models \Phi_\Sigma(C)(a)$ if and only if $\Sigma \models \tilde{\Phi}_\Sigma(C)(a)$, which by Theorem 30 is equivalent to $\Sigma \Vdash_{\mathrm{e}} C(a)$. The second part can be proved analogously. □

7 A System

To check the feasibility of our method in practice, we have implemented a prototype for epistemic querying of OWL Ontologies. The system implements the transformation $\tilde{\Phi}_\Sigma$ of an epistemic concept to its non-epistemic version from Definition 15 involving calls to an underlying standard DL reasoner that offers the reasoning task of instance retrieval. It is implemented on top of the OWL-API[5] extending its classes and interfaces with constructs for epistemic concepts and roles, as shown by the UML class diagram in Figure 1. The new types OWLObjectEpistemicConcept and OWLObjectEpistemicRole are derived from the respective standard types OWLBooleanClassExpression and OWLObjectPropertyExpression to fit the design of the OWL-API.

Using these types, the transformation $\tilde{\Phi}_\Sigma$ is implemented in the class Translator following the visitor pattern mechanism built in the OWL-API. The EQAnswerer uses both a Translator together with an OWLReasoner to perform epistemic reasoning tasks. A running implemented system has been shared on googlecode[6] and can be downloaded for the testing purposes.

[5] http://owlapi.sourceforge.net/
[6] http://code.google.com/p/epistemicdl/

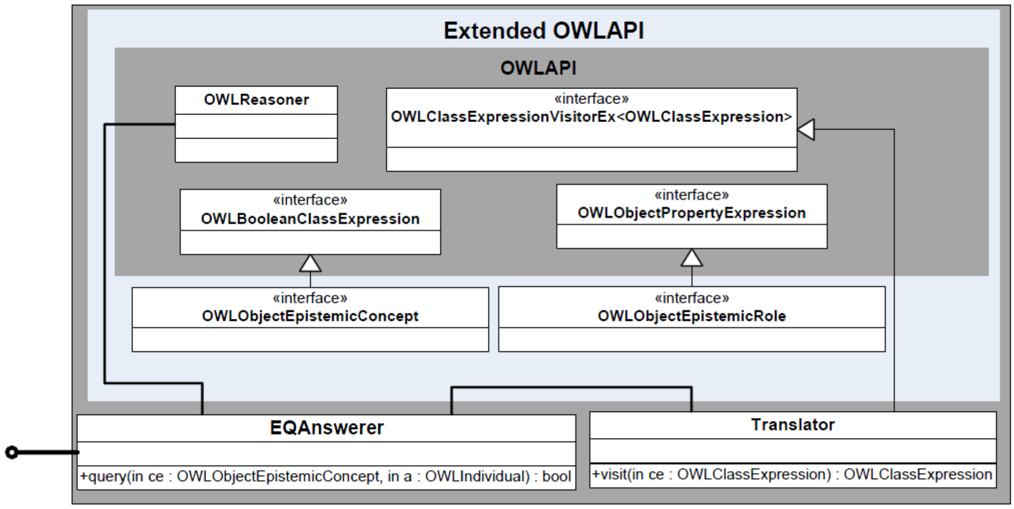

Fig. 1. An overview of the class diagram of the system

For the purpose of testing, we considered the wine ontology[7]. As a measure for our test, we considered the time required to compute the instances of a concept. This suffices as entailment check can not be harder than instance retrieval. We consider different epistemic concepts. For each of these concept , we consider a non-epistemic concept obtained from it by dropping the **K**-operators occurring in it. It is worth mentioning that a direct comparison is debatable as the semantics of the two concepts differ. Nevertheless, we observed that the time required for instance retrieval for a epistemic concept was about one to two order of magnitude higher than that of the concept obtained from it. The performance degrades once ontologies with large number of individuals are considered. The reason for this is that the size of the concept obtained after translating an epistemic concept usually increases with the number of ABox individuals. Yet another observation we made was that the position of **K** in an epistemic concept also effects the overall computation time and the translation time in particular. For example, we noticed that instances retrieval for an epistemic concept where a **K**-operator occurs within the scope of a negation, tends to require much time.

[7] http://www.w3.org/TR/owl-guide/wine.rdf

8 Conclusion and Outlook

We showed that some expressive features of today's DLs such as $\mathcal{SROIQ}$ cause problems when applying the hitherto used semantics to epistemically extended DLs. We suggested a revision to the semantics and proved that this revised semantics solves the aforementioned problem while coinciding with the traditional semantics on less expressive DLs (up to $\mathcal{SRIQ} \setminus U$). Focusing on the new semantics, we provided a way of answering epistemic queries to $\mathcal{SROIQ}$ knowledge bases via a reduction to a series of standard reasoning steps, thereby enabling the deployment of the available highly optimized off-the-shelf DL reasoners. Finally, we presented an implementation allowing for epistemic querying in OWL 2 DL.

Avenues for future research include the following: First, we will investigate to what extent the methods described here can be employed for entailment checks on $\mathcal{SROIQK}$ knowledge bases, i.e., in cases where $\mathbf{K}$ occurs inside the knowledge base. In that case, stronger non-monotonic effects occur and the unique-epistemic-model property is generally lost. On the more practical side, we aim at further developing our initial prototype. We are confident that by applying appropriate optimizations such as caching strategies and syntactic query preprocessing a significant improvement in terms of runtime can be achieved. Moreover, we intend to perform extensive tests with different available OWL reasoners; in our case an efficient handling of (possibly rather extensive) nominal concepts is crucial for a satisfactory performance. In the long run, we aim at demonstrating the added value of epistemic querying by providing an appropriate user-front-end and performing user studies. Furthermore, we will propose an extension of the current OWL standard by epistemic constructs in order to provide a common ground for future applications.

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