A Comparison of Disjunctive Well-founded Semantics

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Abstract. While the stable model semantics, in the form of Answer Set Programming, has become a successful semantics for disjunctive logic programs, a corresponding satisfactory extension of the well-founded semantics to disjunctive programs remains to be found. The many current proposals for such an extension are so diverse, that even a systematic comparison between them is a challenging task. This is mainly caused by the completely different mechanisms applied in the approaches. In order to aid the quest for suitable disjunctive well-founded semantics, we present a systematic approach to a comparison based on level mappings, a recently introduced framework for characterizing logic programming semantics, which was quite successfully used for comparing the major semantics for normal logic programs. We extend this framework to disjunctive logic programs and present alternative characterizations for the strong well-founded semantics (SWFS), the generalized disjunctive well-founded semantics (GDWFS), and the disjunctive well-founded semantics (D-WFS). This will allow us to gain comparative insights into their different handling of negation.

1 Introduction

Two semantics are nowadays considered to be the most important ones for normal logic programs. Stable model semantics [7] is the main two-valued approach (allowing for truth-values *true* and *false*) whereas the major three-valued semantics (adding the value *undefined*) is the well-founded semantics [19]. These two semantics are closely related as shown e.g. in [18]. However, enriching normal logic programs with indefinite information by allowing disjunctions in the head[3] of the clauses separates these two approaches. While disjunctive stable models [15] are a straightforward extension of the stable model semantics, the issue of disjunctive well-founded semantics remains unresolved, although several proposals exist.

[3] For an overview of semantics for disjunctive logic programs we refer to [11] and [13].

Even a comparison of existing proposals is difficult due to the large variety of completely different constructions on which these semantics are based. In [17], Ross introduced the strong well-founded semantics (SWFS) based on a top-down procedure using derivation trees. The generalized disjunctive well-founded semantics (GDWFS) was defined by Baral, Lobo, and Minker in [2], built on several bottom-up operators and the extended generalized closed world assumption [21]. Brass and Dix proposed the disjunctive well-founded semantics (D-WFS) in [4] based on two operators iterating over conditional facts, respectively some general program transformations.

In order to allow for easier comparison of different semantics, a methodology has recently been proposed for uniformly characterizing semantics by means of level mappings, which allow for describing syntactic and semantic dependencies in logic programs [9]. This results in characterizations providing easy comparisons of the corresponding semantics. With the introduction of the framework, normal logic programs have been studied and compared in [9] and [8].

In this paper, we attempt to utilize this approach and present level mapping characterizations for three of the previously mentioned semantics, namely SWFS, GDWFS and D-WFS. The obtained uniform characterizations will allow us to compare the semantics in a new and more structured way. It turns out, however, that even under the uniform level-mapping characterizations the different semantics differ widely, such that there is simply not enough resemblance between the approaches to obtain a coherent picture. We can thus, basically, only confirm in a more formal way what has been known beforehand, namely that the issue of a good definition of well-founded semantics for disjunctive logic programs remains widely open. We still believe that our structured approach delivers structural insights which can help to guide the quest.

The paper is structured as follows. In Section 2, basic notions are presented and we recall shortly the well-founded semantics. Then we devote one section to each of the three semantics recalling the approach itself and presenting the level mapping characterization. We start with SWFS in Section 3, continue with GDWFS in Section 4 and end with D-WFS in Section 5. After that, in Section 6 we compare the characterizations looking for common conditions which might be properties for an appropriate well-founded semantics for disjunctive programs. We conclude with Section 7 and point out further work.

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2 General Notions and Preliminaries

A *disjunctive logic program* Π consists of finitely many universally quantified *clauses* of the form $H_1 \vee \cdots \vee H_l \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ where H_k, A_i, and B_j, for $k = 1, \ldots, l$, $i = 1, \ldots, n$, and $j = 1, \ldots, m$, are atoms of a given first order language, consisting of predicate symbols, function symbols, constants and variables. The symbol $\neg$ is representing default negation. A clause c can be divided into the *head* $H_1 \vee \cdots \vee H_l$ and the *body* $A_1, \ldots, A_n$, $\neg B_1, \cdots, \neg B_m$. If the body is empty then c is called a *fact*. We also abbreviate c by $H \leftarrow A, \neg B$, where H, A and B are sets of pairwise distinct atoms and, likewise, we handle disjunctions D and conjunctions C. A *normal* (*definite*) clause contains exactly one atom in H (no atom in B) and we call a program consisting only of normal (definite) clauses a *normal* (*definite*) logic program. We denote normal programs by P to distinguish from disjunctive ones represented by Π. Any expression is called *ground* if it contains no variables. The *Herbrand base* B_Π is the set of all ground atoms that can be formed by using the given language from Π. A *literal* is either a *positive literal*, respectively an atom, or a *negative literal*, a negated atom, and usually we denote by $A, B, \ldots$ atoms and by $L, M, \ldots$ literals. Moreover, a *disjunction literal* is a disjunction or a negated disjunction. The *extended Herbrand base* EB_Π (*conjunctive Herbrand base* CB_Π) is the set of all disjunctions (conjunctions) that can be formed using pairwise distinct atoms from B_Π. Finally, $\mathsf{ground}(\Pi)$ is the set of all ground instances of clauses in Π with respect to B_Π.

We continue by recalling three-valued semantics based on the truth values true ($\mathbf{t}$), undefined ($\mathbf{u}$), and false ($\mathbf{f}$). A *(partial) three-valued interpretation* I of a normal program P is a set $A \cup \neg B$, for $A, B \subseteq B_P$ and $A \cap B = \emptyset$, where elements in A, B respectively, are $\mathbf{t}$, $\mathbf{f}$, and the remaining wrt. B_P are $\mathbf{u}$. The set of three-valued interpretations is denoted by $I_{P,3}$. Given a three-valued interpretation I, the body of a ground clause $H \leftarrow L_1, \ldots, L_n$ is true in I if and only if $L_i \in I$, $1 \leq i \leq n$, or false in I if and only if $L_i \notin I$ for some i, $1 \leq i \leq n$. Otherwise the body is undefined. The ground clause $H \leftarrow \mathtt{body}$ is true in I if and only if the head H is true in I or $\mathtt{body}$ is false in I or $\mathtt{body}$ is undefined and H is not false in I. Moreover, I is a *three-valued model* for P if and only if all clauses in $\mathsf{ground}(P)$ are true in I. The *knowledge ordering* [6] is recalled which, given two three-valued interpretations I_1 and I_2, is defined as $I_1 \leq_k I_2$ if and only if $I_1 \subseteq I_2$. For a program P and a three-valued interpretation $I \in I_{P,3}$ an *I-partial level mapping* for P is a partial mapping $l : B_P \rightarrow \alpha$ with domain $\mathsf{dom}(l) = \{A \mid A \in I \text{ or } \neg A \in I\}$, where α is some (countable) ordinal. Every such mapping is extended to literals by setting $l(\neg A) = l(A)$ for all $A \in \mathsf{dom}(l)$. Any ordinal α is identified with the set of ordinals β such that $\alpha > \beta$. Thus, any mapping $f : X \rightarrow \{\beta \mid \beta < \alpha\}$ is represented by $f : X \rightarrow \alpha$. Given two ordinals α, β, the lexicographic order $(\alpha \times \beta)$ is also an ordinal with $(a, b) \geq (a', b')$ if and only if $a > a'$ or $a = a'$ and $b \geq b'$ for all $(a, b), (a', b') \in \alpha \times \beta$. This order can be split into its components, namely $(a, b) >_1 (a', b')$ if and only if $a > a'$ for all $(a, b), (a', b') \in \alpha \times \beta$ and $(a, b) \geq_2 (a', b')$ if and only if $a = a'$ and $b \geq b'$ for all $(a, b), (a', b') \in \alpha \times \beta$. Additionally we allow the order $\succ$ which given an ordinal $(\alpha \times \beta)$ is defined as $(a, b) \succ (a', b')$ if and only if $b > b'$ for all (a, b), $(a', b') \in (\alpha \times \beta)$.

We shortly recall the level mapping characterization of the well-founded semantics and refer for the original bottom-up operator to [19].

Definition 2.1. *([9]) Let P be a normal logic program, let I be a model for P, and let l be an I-partial level mapping for P. We say that P* satisfies *(WF)* with respect to I and l *if each $A \in \mathsf{dom}(l)$ satisfies one of the following conditions.*

(WFi) $A \in I$ and there is a clause $A \leftarrow L_1, \ldots, L_n$ in $\mathsf{ground}(P)$ such that $L_i \in I$ and $l(A) > l(L_i)$ for all i.
(WFii) $\neg A \in I$ and for each clause $A \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(P)$ one (at least) of the following conditions holds:
(WFiia) There exists i with $\neg A_i \in I$ and $l(A) \geq l(A_i)$.
(WFiib) There exists j with $B_j \in I$ and $l(A) > l(B_j)$.

If $A \in \mathsf{dom}(l)$ satisfies (WFi), then we say that A *satisfies* (WFi) *with respect to I and l*, and similarly if $A \in \mathsf{dom}(l)$ satisfies (WFii).

Theorem 2.1. *([9]) Let P be a normal logic program with well-founded model M. Then, in the knowledge ordering, M is the greatest model amongst all models I for which there exists an I-partial level mapping l for P such that P satisfies (WF) with respect to I and l.*

Example 2.1. Consider the program $P = \{p \leftarrow \neg q; q \leftarrow q; r \leftarrow \neg p\}$. We obtain the well-founded model $M = \{p, \neg q, \neg r\}$ with $l(p) = 1$, $l(q) = 0$ and $l(r) = 2$. Note that, for $I = \emptyset$ and arbitrary l, P satisfies (WF) wrt. I and l as well but I is not the greatest such model wrt. $\leq_k$ and thus not the well-founded model.

We continue extending some of the previous notions to the disjunctive case. Let I be a set of disjunction literals. The *closure* of I, $\mathsf{cl}(I)$, is the least set I' with $I \subseteq I'$ satisfying the following conditions: if $D \in I'$ then $D' \in I'$ for all D' with $D \subseteq D'$, and for all disjunctions D_1 and D_2, $\neg D_1 \in I'$ and $\neg D_2 \in I'$ if and only if $\neg(D_1 \vee D_2) \in I'$. I is *consistent* if there is no $D \in \mathsf{cl}(I)$ with $\neg D \in \mathsf{cl}(I)$ as well[4]. A *disjunctive three-valued interpretation* I of a disjunctive program Π is a consistent set $A \cup \neg B$, $A, B \subseteq EB_\Pi$, where elements in A are $\mathbf{t}$, elements in B are $\mathbf{f}$, and the remaining wrt. EB_Π are $\mathbf{u}$. The body of a ground clause $H \leftarrow A, \neg B$ is true in I if and only if all literals in the body are true in I, or false in I if and only if there is a D such that either $D \subseteq A$ with $\neg D \in I$ or $D \subseteq B$ with $D \in I$[5]. Otherwise the body is undefined. The truth of a ground clause $H \leftarrow \mathsf{body}$ is identical to normal programs and I is a *disjunctive three-valued model* of Π if every clause in $\mathsf{ground}(\Pi)$ is true in I. The *disjunctive knowledge ordering* $\preceq_k$ is defined as $I_1 \preceq_k I_2$ if and only if $I_1 \subseteq I_2$. Then the corresponding mapping is extended as follows.

Definition 2.2. *For a disjunctive program Π and a disjunctive interpretation I a* disjunctive I-partial level mapping *for Π is a partial mapping $l : EB_\Pi \rightarrow \alpha$ with domain $\mathsf{dom}(l) = \{D \mid D \in I \text{ or } \neg D \in I\}$, where α is some (countable) ordinal. Every such mapping is extended to negated disjunctions by setting $l(\neg D) = l(D)$ for all $D \in EB_\Pi$.*

[4] Here, a consistent set is not automatically closed, in contrast with the assumption made in [17].

[5] The extension is necessary since we might e.g. know the truth of some disjunction without knowing which particular disjunct is true.

Another way of representing disjunctive information are *state-pairs* $A \cup \neg B$, where A is a subset of EB_{Π} such that for all D' if $D \in A$ and $D \subseteq D'$ then $D' \in A$, and B is a subset of CB_{Π} such that for all C' if $C \in B$ and $C \subseteq C'$ then $C' \in B$. Disjunctions in A are **t**, conjunctions in B are **f**, and all remaining are **u**. A state-pair is consistent if whenever $D \in A$ then there is at least one disjunct D' in D such that $D' \notin B$ and whenever $C \in B$ then there is at least one conjunct C' in C such that $C' \notin A$. The notions of models and the disjunctive knowledge ordering can easily be adopted. Note that a state-pair is not necessarily consistent and that it contains indefinite positive and negative information in opposite to disjunctive interpretations where negative information will be precise. Level mappings are adjusted to state-pairs in the following and now we do not extend the mapping to identify $l(D) = l(\neg D)$ since in a state-pair D is a disjunction and $\neg D$ a negated conjunction.

Definition 2.3. *For a disjunctive program Π and a state-pair I a disjunctive I-partial level mapping for Π is a partial mapping $l : (EB_{\Pi} \cup CB_{\Pi}) \to \alpha$ with domain $\mathsf{dom}(l) = \{D \mid D \in I \text{ or } \neg C \in I\}$, where α is some (countable) ordinal.*

3 Strong Well-founded Semantics

We start with SWFS which was introduced by Ross [17] and based on disjunctive interpretations. The derivation rules of the applied top-down procedure are the following. Given a set of disjunction literals I and a disjunctive program Π the *derivate* I' is *strongly derived* from I ($I \Leftarrow I'$) if I contains a disjunction D and $\mathsf{ground}(\Pi)$ a clause $H \leftarrow A_1, \ldots, A_n, \neg B$ such that either

(S1) $H \subseteq D$ and $I' = (I \setminus \{D\}) \cup \{A_1 \vee D, \ldots, A_n \vee D\} \cup \neg B$ or
(S2) $H \not\subseteq D$, $H \cap D \neq \emptyset$, $C = H \setminus D$, and $I' = (I \setminus \{D\}) \cup A \cup \neg B \cup \neg C$.

Consider a ground disjunction D, let $I_0 = \{D\}$ and suppose that $I_0 \Leftarrow I_1 \Leftarrow I_2 \ldots$, then $I_0, I_1, I_2 \ldots$ is a *(strong) derivation sequence* for D. An *active* (strong) derivation sequence for D is a finite derivation sequence for D whose last element, also called a *basis* of D, is either empty or contains only negative literals. A basis $I = \{\neg l_1, \ldots, \neg l_n\}$ is turned into a disjunction $\bar{I} = l_1 \vee \cdots \vee l_n$ and if I is empty, denoting **t**, then $\bar{I}$ denotes **f**. Thus, a *strong global tree* Γ_D^S for a given disjunction $D \in EB_{\Pi}$ contains the root D and its children are all disjunctions of the form $\bar{I}$, where I ranges over all bases for D. The *strong well-founded model* of a disjunctive program Π is called $M_{WF}^S(\Pi)$ and $D \in M_{WF}^S(\Pi)$, i.e. D is true, if some child of D is false and $\neg D \in M_{WF}^S(\Pi)$, i.e. D is false, if every child of D is true. Otherwise, D is undefined and neither D nor $\neg D$ occur in $M_{WF}^S(\Pi)$. In [17], it was shown that $M_{WF}^S(\Pi)$ is a consistent interpretation and that, for normal programs, SWFS coincides with the well-founded semantics[6].

Example 3.1. The following program Π will be used to demonstrate the behavior of the three semantics.

[6] More precisely, the disjunctive model has to be restricted to (non-disjunctive) literals.

$$\begin{array}{rlrl} p \vee q & \leftarrow \neg q & c & \leftarrow \neg l, \neg r \\ q & \leftarrow \neg q & e & \leftarrow \neg f, c \\ b \vee l & \leftarrow \neg r & f & \leftarrow \neg e \\ l \vee r & \leftarrow & g & \leftarrow e \end{array}$$

We obtain a sequence $\{l{\vee}r\} \Leftarrow \{\}$ and $l{\vee}r$ is true as expected. Furthermore, there is a finite sequence in Γ_e^S, namely $\{e\} \Leftarrow \{\neg f, e{\vee}c\} \Leftarrow \{\neg f, \neg l, \neg c\}$ with the only (true) child and e is false. Thus, we have that $M_{WF}^S(\Pi) = \{l{\vee}r, f, \neg b, \neg c, \neg e, \neg g\}$. Literally, this is only a small part of the model and we might close the model (e.g. $\neg(e{\vee}g) \in M_{WF}^S$) for this example, but the strong well-founded is not necessarily closed which does not allow us to add this implicit information in general.

The level mapping framework is based on bottom-up operators and SWFS is a top-down-procedure so we introduced a bottom-up operator on derivation trees defined on $\boldsymbol{\Gamma_\Pi^S}$ which is the power set of Γ_Π^S, i.e. the set of all strong global trees with respect to Π.

Definition 3.1. *Let Π be a disjunctive logic program, $M_{WF}^S(\Pi)$ the strong well-founded model, and $\Gamma \in \boldsymbol{\Gamma_\Pi^S}$. We define:*

- ${T'}_\Pi^S(\Gamma) = \{\Gamma_D^S \in \Gamma_\Pi^S \mid \Gamma_D^S$ *contains an active strong derivation sequence with child C where $\neg C \in M_{WF}^S(\Pi)$ and $\Gamma_C^S \in \Gamma$ if $C \neq \{\}\}$*
- $U_\Pi^S(\Gamma) = \{\Gamma_D^S \in \Gamma_\Pi^S \mid$ *for all active strong derivation sequences in Γ_D^S the corresponding child C is true in $M_{WF}^S(\Pi)$ and $\Gamma_C^S \in \Gamma\}$*
- ${W'}_\Pi^S(\Gamma) = {T'}_\Pi^S(\Gamma) \cup U_\Pi^S(\Gamma)$
- ${W'}_\Pi^S \uparrow 0 = \emptyset$, ${W'}_\Pi^S \uparrow n{+}1 = {W'}_\Pi^S({W'}_\Pi^S \uparrow n)$ *and* ${W'}_\Pi^S \uparrow \alpha = \bigcup_{\beta<\alpha} {W'}_\Pi^S \uparrow \beta$ *for limit ordinal α*

This operator yields the assignment of the stage which slightly differs from recursive definition introduced in [17] and we use only as the stage to avoid ambiguities.

Definition 3.2. *Let Π be a disjunctive logic program and the* stage *s be a partial function $s : EB_\Pi \to \alpha$ for some ordinal α. Let $D \in M_{WF}^S$ or $\neg D \in M_{WF}^S$. Then $s(D) = \alpha$ where α is the least ordinal such that $\Gamma_D^S \in ({W'}_\Pi^S \uparrow (\alpha + 1))$.*

It is obvious, by Definition 3.2, that any disjunction D with assigned stage occurs in M_{WF}^S. The contrary will be shown by means of Definition 3.1.

Lemma 3.1. *Let Π be a disjunctive logic program and $D \in EB_\Pi$. If D is true or false in M_{WF}^S then $D \in \mathsf{dom}(s)$.*

Proof. We have to show that whenever D is true or false in M_{WF}^S then $\Gamma_D^S \in ({W'}_\Pi^S \uparrow \alpha)$ for some α. Assume without loss of generality that there is a $D \in M_{WF}^S$ with $\Gamma_D^S \notin ({W'}_\Pi^S \uparrow \alpha)$ for all α. Since $D \in M_{WF}^S$, we know that there is an active strong derivation sequence in Γ_D^S with child C and $\neg C \in M_{WF}^S$. By Definition of ${T'}_\Pi^S$ and since $\Gamma_D^S \notin ({W'}_\Pi^S \uparrow \alpha)$ for any α we know that

$\Gamma_C^S \notin W_{WF}^S \uparrow \alpha$ for any α. The same holds for any child of D we choose one of these, i.e. C. Since $\neg C \in M_{WF}^S$, we know that all active strong derivation sequences in Γ_D^S have a true child. By Definition of U_Π^S and since $\Gamma_C^S \notin ({W'}_\Pi^S \uparrow \alpha)$ for any α we know that there is at least one true child C_1 with $\Gamma_{C_1}^S \notin {W'}_\Pi^S \uparrow \alpha$ for any α. We can now apply the argument from D also to C_1 and obtain again a false child like C beforehand. In this manner we obtain a transfinite sequence of children and none of their corresponding trees are in ${W'}_\Pi^S \uparrow \alpha$ for some α. But then, by means of the recursive definition of truth in derivation trees the truth values are not defined in this transfinite sequence of children which contradicts our initial assumption. ■

Even though the stage is defined for all disjunctions which are true or false in the strong well-founded model, it is not the desired result for the alternative characterization.

Example 3.2.

$$
\begin{aligned}
p &\leftarrow \\
q &\leftarrow p \\
r &\leftarrow r \\
s &\leftarrow \neg r \\
t &\leftarrow \neg s
\end{aligned}
$$

Γ_p^S has only one empty child, i.e. $\Gamma_p^S \in {W'}_\Pi^S \uparrow 1$ and $s(p) = 0$. Obviously, Γ_q^S also only has that child and $s(q) = 0$ as well. Γ_r^S has no children and we set the stage of r to 0. Then, Γ_s^S has only one false child, r, i.e. is of stage 1 and similarly Γ_t^S has only one true child s, thus $s(t) = 2$.
Since this program is normal, we may also apply the level mapping characterization of the well-founded semantics and obtain also $l(r) = 0$, $l(s) = 1$, and $l(t) = 2$. But $l(p) = 0$ and $l(q) = 1$ by Theorem 2.1 and (WFi) of Definition 2.1 and we obtain the dependency between p and q given by the second clause which is lost in the stage assignment.

We thus prefer to have a characterization which for normal programs also coincides with the characterization of the well-founded semantics. Nevertheless, the stage can be used to prove certain properties of the strong well-founded model starting with the following lemma.

Lemma 3.2. *Let Π be a disjunctive logic program and $D, D' \in EB_\Pi$ with $D' \subseteq D$ and $D'' = D \setminus D'$. If $\Gamma_{D'}^S$ contains an active strong derivation sequence with child D'_1 then Γ_D^S also contains an active strong derivation sequence with child D_1 such that $D_1 \subseteq D'_1$.*

Proof. Consider the active strong derivation sequence $\{D'\}, I'_1, \ldots, I'_r$ where $H \leftarrow A_1, \ldots A_n, \neg B_1, \ldots, \neg B_m$ is the clause c in $\mathsf{ground}(\Pi)$ which is used for the first derivation $\{D'\} \Leftarrow I'_1$ of that sequence. We consider two cases.

We apply (S1). Then $H \subseteq D'$ and $I'_1 = \{A_1 \vee D', \ldots, A_n \vee D', \neg B_1, \ldots, \neg B_m\}$. But then we also have a derivation $\{D\} \Leftarrow I_1$ with $I_1 = \{A_1 \vee D, \ldots, A_n \vee D, \neg B_1, \ldots, \neg B_m\}$ by (S1) since $H \subseteq D' \subseteq D$. If there is no atom A_i, $1 \leq i \leq n$, occurring in c then we already obtained the basis and in both cases the child is $B_1 \vee \ldots \vee B_m$, i.e. $D_1 = D'_1$, and thus $D_1 \subseteq D'_1$. Otherwise, there is at least one atom A_i in c such that we have $A_i \vee D'$ in I'_1 and $A_i \vee D$ in I_1 with $(\{A_i\} \cup D') \subseteq (\{A_i\} \cup D)$. Nevertheless, the sets of negative atoms obtained in the first derivation step are identical, i.e. $B_1 = B'_1$ and the sets of additional negated atoms C_1 and C'_1 are empty since we applied (S1) where the index, in this case 1, represents the derivation step.

Alternatively, we apply (S2). Then $H \not\subseteq D'$, $H \cap D' \neq \emptyset$, $C' = H \setminus D'$, and $I'_1 = \{A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m\} \cup \neg C'$. Since $D' \subseteq D$ we also have $H \cap D \neq \emptyset$.
(1) If $H \not\subseteq D$ then by (S2) we also have a derivation $\{D\} \Leftarrow I_1$ with $C = H \setminus D$ and $I_1 = \{A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m\} \cup \neg C$. If $D'' \cap H = \emptyset$ then $C' = C$. If $D'' \cap H \neq \emptyset$ then $C' = C \cup (D'' \cap H)$.
(2) Otherwise, $H \subseteq D$ and we can apply (S1) instead, yielding $\{D\} \Leftarrow I_1$ with $I_1 = \{A_1 \vee D, \ldots, A_n \vee D, \neg B_1, \ldots, \neg B_m\}$.
Again, if c does not contain any A_i, $1 \leq i \leq n$, then I'_1 is a basis and $B \cup C'$ is a child in $\Gamma^S_{D'}$, and thus $B \cup C$, respectively B in case of (S1), is a child in Γ^S_D. Then $B(\cup C) \subseteq B \cup C'$. Otherwise, there is at least one A_i in c such that we have A_i in I'_1 and A_i in I_1, respectively $A_i \vee D$ in I_1 (depending on whether we applied (S1) or (S2)), with $\{A_i\} \subseteq \{A_i\}$, respectively $\{A_i\} \subseteq (\{A_i\} \cup D)$, where $B_1(\cup C_1) \subseteq B_1 \cup C'_1$. Note that the now introduced additional indices refer to the derivation step.
As we have seen, no matter whether we apply (S1) or (S2), for each resulting positive disjunction in I'_1 there is also a positive disjunction in I_1 which subsumes the one from I'_1 which allows us to apply the same argument also to the following derivation steps of the active strong derivation sequence $I'_1, \ldots, I'_r$. We obtain the active strong derivation sequence $\{D\}, I_1, \ldots, I_r$ in Γ^S_D and that $B_q(\cup C_q) \subseteq B_q \cup C'_q$ holds for each $1 \leq q \leq r$. But then $\bigcup_{q=1\ldots r} B_q(\cup C_q) \subseteq \bigcup_{q=1\ldots r} B_q \cup C'_q$ which corresponds to $D_1 \subseteq D'_1$. ■

We use this lemma to show that whenever we know that a disjunction D is true with a certain stage then all disjunctions containing D are also true with at most the same stage, respectively whenever we know that a disjunction D is false with a stage $s(D)$ then all subdisjunctions of D are also false with a stage smaller or equal to $s(D)$.

Lemma 3.3. *Let $D \in EB_\Pi$ with $s(D) = \alpha$.*

1. *If $D \in M^S_{WF}$ then $D' \in M^S_{WF}$ and $s(D') \leq \alpha$ for all disjunctions D' with $D \subseteq D'$.*
2. *If $\neg D \in M^S_{WF}$ then $\neg D' \in M^S_{WF}$ and $s(D') \leq \alpha$ for all disjunctions D' with $D' \subseteq D$.*

Proof. We are going to prove the two statements by one transfinite induction on the stage of D.

Let $s(D) = 0$. If $D \in M_{WF}^S$ then there is at least one child in Γ_D^S which is false. By Definition 3.2, $\Gamma_D^S \in W'^S_\Pi \uparrow 1$ and thus, by Definition 3.1, this child is empty. Then, by Lemma 3.2, for any D' with $D \subseteq D'$ there is an active strong derivation sequence in $\Gamma_{D'}^S$ with child C' where $C' \subseteq C$, i.e. $C' = \{\}$. Therefore, D' also has an empty (false) child and $\Gamma_{D'}^S \in W'^S_\Pi \uparrow 1$, i.e. $s(D') = 0$ and $s(D') \leq s(D)$.

Alternatively $\neg D \in M_{WF}^S$ and all children in Γ_D^S are true. Since, by Definition 3.2, $\Gamma_D^S \in W'^S_\Pi \uparrow 1$ we know that there are no children at all in Γ_D^S. Assume that there is an active strong derivation sequence in $\Gamma_{D'}^S$ with child D'_1 for any D' with $D' \subseteq D$. By Lemma 3.2, we then have that Γ_D^S also contains an active strong derivation sequence with child D_1 which contradicts the assumption. Hence, $\Gamma_{D'}^S$ has no children, $\neg D' \in M_{WF}^S$, and $s(D') = 0$.

Suppose that the lemma holds for all disjunctions C with $s(C) = \beta$, $\beta < \alpha$, i.e. if $C \in M_{WF}^S$ then $C' \in M_{WF}^S$ and $s(C') \leq \beta$ for all disjunctions C' with $C \subseteq C'$, and if $\neg C \in M_{WF}^S$ then $\neg C' \in M_{WF}^S$ and $s(C') \leq \beta$ for all disjunctions C' with $C' \subseteq C$. Let $s(D) = \alpha$. We have to consider two cases.
(1) If $D \in M_{WF}^S$ then there is at least one child in Γ_D^S which is false. By Definition 3.2, we know that $\Gamma_D^S \in W'^S_\Pi \uparrow (\alpha+1)$. Consider the corresponding active strong derivation sequence of a false child C with $\Gamma_C^S \in W'^S_\Pi \uparrow \alpha$, i.e. $s(C) < \alpha$ say $s(C) = \beta$. By Lemma 3.2, for any D' with $D \subseteq D'$ we know that there also is an active strong derivation sequence with child C' such that $C' \subseteq C$. By induction hypothesis we have $\neg C' \in M_{WF}^S$ and $s(C') \leq \beta$. Thus $\Gamma_{C'}^S \in W'^S_\Pi \uparrow (\beta+1)$ and $\beta + 1 \leq \alpha$. Hence, $\Gamma_{C'}^S \in W'^S_\Pi \uparrow (\alpha+1)$ $s(D') \leq \alpha$ by Definition 3.2.
(2) If $\neg D \in M_{WF}^S$ then all children in Γ_D^S are true. Consider any disjunction D' with $D' \subseteq D$. If $\Gamma_{D'}^S$ does not contain any active derivation sequences then $s(D') = 0$, $0 \leq \alpha$ and $\neg D' \in M_{WF}^S$. Thus consider alternatively any active strong derivation sequence with child C'. Then, by Lemma 3.2, Γ_D^S also contains an active strong derivation sequence with child C such that $C \subseteq C'$. We know that $\Gamma_D^S \in W'^S_\Pi \uparrow (\alpha+1)$ by Definition 3.2. Thus, by Definition 3.1, for child C we have $\Gamma_C^S \in W'^S_\Pi \uparrow \alpha$, i.e. $s(C) < \alpha$ say $s(C) = \beta$. Thus, by induction hypothesis and $C \subseteq C'$, $C' \in M_{WF}^S$ and $s(C') \leq \beta$. Since we have shown for any arbitrary child C' that it is true with stage less than or equal to β, we know that the trees of all children of D' are contained in $W'^S_\Pi \uparrow \beta+1$ with $\beta+1 \leq \alpha$. Then, by Definition 3.1, $\Gamma_{D'}^S \in W'^S_\Pi \uparrow (\alpha+1)$ and $s(D') \leq \alpha$. ■

Example 3.3. We demonstrate this with the given program Π.

$$\begin{aligned} p &\leftarrow \quad p \\ q &\leftarrow \neg s, \neg t \\ s \vee t &\leftarrow \end{aligned}$$

We know that $p \vee q$ only has one child $s \vee t$ which is true with stage 0. Thus $p \vee q$ is false with $s(p \vee q) = 1$. For the same reason, q is false with $s(q) = 1$. The stage of a subdisjunction does not have to be equal: Γ_p^S does not have any children and thus p is false but with stage 0.

Lemma 3.3 shows that the strong well-founded model satisfies the first and one direction of the second condition of the closure of a set of disjunction literals. However, for showing that it is in fact closed we also need to show the other direction of the second condition of this definition, which appears to be rather difficult.
Table 1 shows our current knowledge about the assignment of truth values in the strong well-founded semantics for a disjunction $(p \vee q)$ given the values of its two disjuncts p and q, respectively the truth value of a disjunct s given the truth value of the disjunction $(r \vee s)$ and the other disjunct r. The three entries 'n.a.' stand for 'not allowed' because if we e.g. already know that r is true then $(r \vee s)$ cannot be undefined by Lemma 3.3. The truth values in parentheses with question mark are not intended but we did not prove yet that they cannot occur.

Table 1. Truth values in the strong well-founded semantics

p	q	$(p \vee q)$	r	$(r \vee s)$	s
f	**f**	**f** (/**u**?^{a})	**f**	**f**	**f**
f	**u**	**u** (/**t**?^{d})	**f**	**u**	**u** (/**f**?^{c})
f	**t**	**t**	**f**	**t**	**t** (/**u**?^{b})
u	**f**	**u** (/**t**?^{d})	**u**	**f**	n.a.
u	**u**	**u**/**t**	**u**	**u**	**f**/**u**
u	**t**	**t**	**u**	**t**	**u**/**t** (/**f**?^{e})
t	**f**	**t**	**t**	**f**	n.a.
t	**u**	**t**	**t**	**u**	n.a.
t	**t**	**t**	**t**	**t**	**f**/**u**/**t**

The first thing we see is that the assignment of the truth value to a disjunction is not functional. Given two undefined disjuncts the disjunction may be true or undefined. As an example consider the program just consisting of $p \vee q \leftarrow$. Since p has q as the only child and q has only one child p and this alternates forever, both, p and q, are undefined. Nevertheless, $(p \vee q)$ has one empty child and is thus true. Alternatively, consider the program with the two clauses $p \leftarrow \neg p$ and $q \leftarrow \neg q$. Then p and q are both undefined again but now $(p \vee q)$ is also undefined since it has now only the two children p and q. This example also shows that q may be undefined if we know that p and $(p \vee q)$ are undefined, and if we drop the second clause then q may also be false in this case.

Let us now consider the cases where not intended results remain to be removed. If p and q are false then $(p \vee q)$ cannot be true by consistency of the strong well-founded model but it may be undefined (a). If we can show that

$(p \lor q)$ has to be false then s cannot be false (c) knowing that r is false and $(r \lor s)$ is undefined. If r is false and $(r \lor s)$ is true then s cannot be false by consistency however it may be undefined (b). But if we can show that it has to be true in general then $(p \lor q)$ cannot be true given a false and an undefined disjunct (d) and likewise given a true disjunction $(r \lor s)$ and r undefined then s cannot be false (e).
Showing one of these properties is most likely done by an transfinite induction over the stage similar to Lemma 3.3. Since undefined disjunctions cannot have a value for the stage it is most reasonable trying to prove that the cases (a) and (b) cannot occur because all assumptions in both cases have a defined stage. However, the following example will show that even that is rather complicated.

Example 3.4. Let Π be a disjunctive logic program and let the following clauses be all clauses in $\mathsf{ground}(\Pi)$ which contain p, q, a, or b in the head.

$$\begin{aligned} p \lor q &\leftarrow a, \neg s \\ p \lor q &\leftarrow b, \neg t \\ a \lor b &\leftarrow \neg r \end{aligned}$$

The disjunction $(p \lor q)$ has three different children. We may apply (S1) with the first clause and (S2) with the third clause and obtain $\{p \lor q\} \Leftarrow \{p \lor q \lor a, \neg s\} \Leftarrow \{\neg b, \neg r, \neg s\}$. Or we apply (S1) with the second clause and (S2) with the third clause and obtain $\{p \lor q\} \Leftarrow \{p \lor q \lor b, \neg t\} \Leftarrow \{\neg a, \neg r, \neg t\}$. Alternatively, we can also apply (S1) with the first and the second clause (in arbitrary order) and then (S1) with the third clause and obtain e.g. $\{p \lor q\} \Leftarrow \{p \lor q \lor a, \neg t\} \Leftarrow \{p \lor q \lor a \lor b, \neg t, \neg s\} \Leftarrow \{\neg r, \neg s, \neg t\}$. Thus the three children are $(b \lor r \lor s)$, $(a \lor r \lor t)$, and $(r \lor s \lor t)$.
For p, and symmetrically for q, we only have two children. We may apply (S2) with the first clause and (S2) with the third clause and obtain $\{p\} \Leftarrow \{a, \neg s, \neg q\} \Leftarrow \{\neg b, \neg r, \neg s, \neg q\}$. Or we apply (S2) with the second clause and (S2) with the third clause and obtain $\{p\} \Leftarrow \{b, \neg t, \neg q\} \Leftarrow \{\neg a, \neg r, \neg t, \neg q\}$. So we have two children $(b \lor r \lor s \lor q)$ and $(a \lor r \lor t \lor q)$ for p and two children $(b \lor r \lor s \lor p)$ and $(a \lor r \lor t \lor p)$ for q.

Assume we want to show in general that if p and q are false then also $(p \lor q)$ is false using an inductive argument over the stage and assume that we have shown that if $p' \lor q'$ is true and p' is false then q' has to be true. Since p and q are false, all children of them are true. Since q is false and $(b \lor r \lor s \lor q)$ is a true child of p we know that $(b \lor r \lor s)$ has to be true by our additional assumption. This allows us to derive that two of the children of $(p \lor q)$ have to be true. Unfortunately, this argument cannot be applied to the third child $(r \lor s \lor t)$ and it is not clear at all how it could be proven that $(r \lor s \lor t)$ is true. One attempt could be to show that a or b is false as well. Then $(r \lor s)$ or $(r \lor t)$ had to be true by our assumption and thus also $(r \lor s \lor t)$ by Lemma 3.3. But there is no evident argument stating that a or b should be false, both may also be undefined.

If we want to prove that q is true given that $(p \lor q)$ is true and p is false the problem is the very same. One of the children of $(p \lor q)$ has to be false and we assume that if p' and q' are false that $(p' \vee q')$ is false as well. If $(b \vee r \vee s)$ or $(a \vee r \vee t)$ is the false child then by our assumption $(b \vee r \vee s \vee p)$ or $(a \vee r \vee t \vee p)$ is also false since p is false. Otherwise $(r \vee s \vee t)$ is the false child and thus all disjuncts are false by Lemma 3.3. Then in this case we need to show that a or b is false for showing that there is a false child for q but it also is not clear at all how this could be done.

The example displays that a general proof that the intended properties from Table 1 hold can be expected to be rather difficult. So far, we have not been able to come up with either a proof nor a counterexample, and so we leave this as an open problem. This will not influence the subsequent discussion, but we will make sure that we remark whenever this open problem comes into play.

Even though we could not prove that all desired properties hold for the strong well-founded semantics, we may use the results from Lemma 3.3 to represent the strong well-founded model by a minimal set: whenever a disjunction is contained in the strong well-founded model then any superset of that disjunction is contained implicitly as well and, likewise, whenever a negated disjunction occurs then implicitly any subset of the disjunction occurs negated as well.

We already have seen that the stage is not a desired result for an alternative characterization of the strong well-founded semantics. In the following we modify the operator given in Definition 3.1.

Definition 3.3. *Let Π be a disjunctive logic program, $M^S_{WF}(\Pi)$ the strong well-founded model, and $\Gamma \in \boldsymbol{\Gamma^S_\Pi}$. We define:*

- $T^S_\Pi(\Gamma) = \{\Gamma^S_D \in \Gamma^S_\Pi \mid \Gamma^S_D$ *contains an active strong derivation sequence* $\{D\}, I_1, \ldots, I_r$ *with child* $C = \bar{I}_r$ *and* $I_1 = \{D_1, \ldots, D_n, \neg D_{n+1}, \ldots, \neg D_m\}$ *where* $\neg C \in M^S_{WF}(\Pi)$, $\Gamma^S_C \in \Gamma$ *if* $C \neq \{\}$, $\Gamma^S_{D_i} \in \Gamma$, $D_i \in M^S_{WF}$, $\Gamma^S_{D_j} \in \Gamma$, $\neg D_j \in M^S_{WF}$ *for all* $i = 1, \ldots, n$ *and* $j = n+1, \ldots, m\}$
- $U^S_\Pi(\Gamma) = \{\Gamma^S_D \in \Gamma^S_\Pi \mid$ *for all active strong derivation sequences in* Γ^S_D *the corresponding child* C *is true in* $M^S_{WF}(\Pi)$ *and* $\Gamma^S_C \in \Gamma\}$

The information is joined by $W^S_\Pi(\Gamma) = T^S_\Pi(\Gamma) \cup U^S_\Pi(\Gamma)$ and iterated: $W^S_\Pi \uparrow 0 = \emptyset$, $W^S_\Pi \uparrow n+1 = W^S_\Pi(W^S_\Pi \uparrow n)$ and $W^S_\Pi \uparrow \alpha = \bigcup_{\beta<\alpha} W^S_\Pi \uparrow \beta$ for limit ordinal α. If we could have shown that all the not intended properties given in Table 1 do not hold then it should not be necessary to mention the child C in case of T^S_Π and the additional conditions related to it. It should be possible to show that all the true elements in the first derivate already imply that.

Example 3.5. Reconsider the program Π from Example 3.2

$$
\begin{aligned}
p &\leftarrow \\
q &\leftarrow p \\
r &\leftarrow r \\
s &\leftarrow \neg r \\
t &\leftarrow \neg s
\end{aligned}
$$

Using the minimal set representation, the strong well-founded model M_{WF}^{S} is $\{p, q, s, \neg(r \vee t)\}$. We have $W_{\Pi}^{S} \uparrow 0 = \emptyset$ and then $W_{\Pi}^{S} \uparrow 1 = W_{\Pi}^{S}(\emptyset) = T_{\Pi}^{S}(\emptyset) \cup U_{\Pi}^{S}(\emptyset)$ and $T_{\Pi}^{S}(\emptyset) = \{\Gamma_{p}^{S}\}$ because Γ_{p}^{S} has only one empty child and the first derivate in the corresponding sequence is empty. We also use here the minimal representation because any tree $\Gamma_{p \vee D}^{S}$ for some disjunction D contains the very same active strong derivation sequence. $U_{\Pi}^{S}(\emptyset) = \{\Gamma_{r}^{S}\}$ because there is no active strong derivation sequence in Γ_{r}^{S} and $W_{\Pi}^{S} \uparrow 1 = \{\Gamma_{p}^{S}, \Gamma_{r}^{S}\}$. Then we have $W_{\Pi}^{S} \uparrow 2 = \{\Gamma_{p}^{S}, \Gamma_{r}^{S}, \Gamma_{q}^{S}, \Gamma_{s}^{S}\}$ and finally $W_{\Pi}^{S} \uparrow 3 = \{\Gamma_{p}^{S}, \Gamma_{r}^{S}, \Gamma_{q}^{S}, \Gamma_{s}^{S}, \Gamma_{t}^{S}, \Gamma_{(r \vee t)}^{S}\}$ which also is the least fixed point for this example. We can see that this also includes implicitly all trees whose roots are disjunctions containing at least one element from the minimal set representation of M_{WF}^{S}, i.e. in this case the least fixed point is in fact Γ_{Π}^{S}. Moreover, we now keep the dependency between p and q since Γ_{q}^{S} only appears in $W_{\Pi}^{S} \uparrow 2$ after we know that Γ_{p}^{S}, respectively $\Gamma_{p \vee q}^{S}$, is contained in $W_{\Pi}^{S} \uparrow 1$.

We will now show that this operator is monotonic.

Proposition 3.1. *Given a logic program Π and the strong well-founded model $M_{WF}^{S}(\Pi)$, the operator W_{Π}^{S} is monotonic.*

Proof. Let $\Gamma_1, \Gamma_2 \in \boldsymbol{\Gamma_{\Pi}^{S}}$, $\Gamma_1 \subseteq \Gamma_2$, and $\Gamma_{D}^{S} \in W_{\Pi}^{S}(\Gamma_1)$. We have to show that $\Gamma_{D}^{S} \in W_{\Pi}^{S}(\Gamma_2)$ as well.
(1) Consider $\Gamma_{D}^{S} \in T_{\Pi}^{S}(\Gamma_1)$. Then Γ_{D}^{S} contains an active strong derivation sequence $\{D\}, I_1, \ldots, I_r$ with child $C = \bar{I}_r$ and $I_1 = \{D_1, \ldots, D_n, \neg D_1, \ldots, \neg D_m\}$ where $\neg C \in M_{WF}^{S}(\Pi)$, $\Gamma_{C}^{S} \in \Gamma$ if $C \neq \{\}$, $\Gamma_{D_i}^{S} \in \Gamma$, $D_i \in M_{WF}^{S}$, $\Gamma_{D_j}^{S} \in \Gamma$, $\neg D_j \in M_{WF}^{S}$, for all $i = 1, \ldots, n$ and $j = 1, \ldots, m\}$. Since $\Gamma_1 \subseteq \Gamma_2$ we know that also $\Gamma_{C}^{S} \in \Gamma_2$ if $C \neq \{\}$, $\Gamma_{D_i}^{S} \in \Gamma_2$, and $\Gamma_{D_j}^{S} \in \Gamma_2$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Hence $\Gamma_{D}^{S} \in T_{\Pi}^{S}(\Gamma_2)$ and thus $\Gamma_{D}^{S} \in W_{\Pi}^{S}(\Gamma_2)$.
(2) Alternatively, suppose that $\Gamma_{D}^{S} \in U_{\Pi}^{S}(\Gamma_1)$. Then for all active strong derivation sequences in Γ_{D}^{S} the corresponding child C is true in $M_{WF}^{S}(\Pi)$ and $\Gamma_{C}^{S} \in \Gamma_1$. Again, since $\Gamma_1 \subseteq \Gamma_2$, for all children C, we have $\Gamma_{C}^{S} \in \Gamma_2$. Hence $\Gamma_{D}^{S} \in U_{\Pi}^{S}(\Gamma_2)$ and thus $\Gamma_{D}^{S} \in W_{\Pi}^{S}(\Gamma_2)$. ∎

Thus we can apply the Tarski fixed-point theorem which yields that the operator W_{Π}^{S} always has a least fixed point. This least fixed point coincides with M_{WF}^{S} in so far that whenever a tree Γ_{D}^{S} is contained in the least fixed point then D or $\neg D$ is contained in the strong well-founded model which is a straightforward consequence of the definition of W_{Π}^{S}. Before we show the other direction we present a property necessary for that.

Lemma 3.4. *Let Π be a disjunctive logic program and $\{D\}, I_1, \ldots, I_r$ be an active strong derivation sequence in Γ_{D}^{S} with child $C = \bar{I}_r$ such that $\neg C \in M_{WF}^{S}$. For all q, $1 \leq q \leq r$, if $D' \in I_q$ then $D' \in M_{WF}^{S}$ and if $\neg D' \in I_q$ then $\neg D' \in M_{WF}^{S}$.*

Proof. We are going to prove that lemma for each derivate starting with the basis I_r. The basis does not contain any positive disjunctions so we only consider $\neg D' \in I_r$. Then $D' \in C$. We know that $\neg C \in M^S_{WF}$ and thus, by Lemma 3.3, that $\neg D' \in M^S_{WF}$.
Assume we have shown for I_q, $q \le r$, that if $D' \in I_q$ then $D' \in M^S_{WF}$ and if $\neg D' \in I_q$ then $\neg D' \in M^S_{WF}$. We show that the claim holds for the derivate I_{q-1}. Let $\neg D'$ be in I_{q-1} where D' is an arbitrary atom. No rule is applicable to $\neg D'$ and thus it occurs unchanged in I_q. Then $\neg D' \in M^S_{WF}$ by assumption. Alternatively, let D' be in I_{q-1} where D' is an arbitrary disjunction. If D' is not the disjunction used in the derivation step $I_{q-1} \Leftarrow I_q$ then it occurs unchanged in I_q which by assumption means $D' \in M^S_{WF}$. So let D' be the disjunction used for the derivation step $I_{q-1} \Leftarrow I_q$. But then we can construct an active strong derivation sequence in $\Gamma^S_{D'}$ with a false child by choosing the very same clause and application rule like in $I_{q-1} \Leftarrow I_q$ for the first derivation step $\{D'\} \Leftarrow I'_1$. All elements of I'_1 occur also in I_q and we thus construct the remaining sequence by applying to each positive disjunction in I'_1 the same rule and clause as in the sequence $\{D\}, I_1, \ldots, I_r$, and likewise for each thereby obtained positive disjunction in any $I_{q'}$, $q' > 1$. We obtain the sequence $\{D'\}, I'_1, \ldots, I'_{r'}$ and $I'_{r'} \subseteq I_r$ since any negative literal obtained in any derivation step also occurs in I_r by construction. Then $\neg \bar{I}'_{r'} \in M^S_{WF}$, $\Gamma^S_{D'}$ contains a false child and thus $D' \in M^S_{WF}$. ∎

Unfortunately, there is no similar property for active strong derivation sequences with true children. Recall Example 3.3. We have shown that q is false because there is an active strong derivation sequence with true child $(s \vee t)$. But e.g. the basis contains $\neg s$ and $\neg t$ and both are undefined in the strong well-founded model.

We now show that whenever a disjunction is true or false in the strong well-founded model then its tree is also contained in the least fixed point of W^S_Π.

Proposition 3.2. *Let Π be a disjunctive logic program. If $D \in M^S_{WF}$ or $\neg D \in M^S_{WF}$ then Γ^S_D is contained in $\mathsf{lfp}(W^S_\Pi)$.*

Proof. Suppose that $D \in M^S_{WF}$ or $\neg D \in M^S_{WF}$. By Lemma 3.1, we know that the stage $s(D)$ is defined, thus we are going to prove by transfinite induction on $s(D)$ that Γ^S_D is contained in $\mathsf{lfp}(W^S_\Pi)$.

Let $s(D)$ be 0. By Definition 3.2, $\Gamma^S_D \in {W'}^S_\Pi \uparrow 1$ and we consider two cases.
(1) If $D \in M^S_{WF}$ then Γ^S_D contains at least one false child and, in fact, by Definition 3.1, there is an empty child. Thus, $\Gamma^S_D \in W^S_\Pi(\emptyset)$, i.e. $\Gamma^S_D \in W^S_\Pi \uparrow 1$ and therefore $\Gamma^S_D \in \mathsf{lfp}(W^S_\Pi)$.
(2) If $\neg D \in M^S_{WF}$ then, by Definition 3.1, there are no children at all in Γ^S_D. Then $\Gamma^S_D \in U^S_\Pi(\emptyset)$, $\Gamma^S_D \in W^S_\Pi(\emptyset)$ and $\Gamma^S_D \in \mathsf{lfp}(W^S_\Pi)$.

Suppose for all C with $s(C) < \alpha$ that $\Gamma^S_C \in \mathsf{lfp}(W^S_\Pi)$. Let $s(D) = \alpha$. We have to consider two cases.
(1) Let $\neg D \in M^S_{WF}$. We know that for each active strong derivation sequence in Γ^S_D the corresponding child C is true. By Definition 3.1, the stage of any C

has to be smaller than $s(D)$, thus $s(C) < \alpha$. Then, by induction hypothesis, $\Gamma_C^S \in \mathsf{lfp}(W_\Pi^S)$ for all children C, and hence, by Definition of U_Π^S, $\Gamma_D^S \in \mathsf{lfp}(W_\Pi^S)$.
(2) Let $D \in M_{WF}^S$. We know that there is at least one active strong derivation sequence where the corresponding child C is false. By Definition 3.1, the stage of the child C has to be smaller than $s(D)$, i.e.$s(C) < \alpha$. Then, by induction hypothesis, $\Gamma_C^S \in \mathsf{lfp}(W_\Pi^S)$ for all false children C with minimal stage. Let $\{D\}, I_1, \ldots, I_r$ be an active strong strong derivation sequence with child C of minimal stage where I_r is the basis. We are going to prove for all $D' \in I_q$, respectively $\neg D' \in I_q$, $0 \leq q \leq r$ where $I_0 = \{D\}$, that $\Gamma_{D'}^S \in \mathsf{lfp}(W_\Pi^S)$. This will finish the proof, because D is the only disjunction occurring in I_0.

Let $q = r$. Since I_r is the basis there is no positive disjunction in I_r we only have to consider $\neg D' \in I_r$. Since $D' \in C$ and $\neg C \in M_{WF}^S$, by Lemma 3.3, we know that $\neg D' \in M_{WF}^S$ and $s(D') \leq \beta$. Then $\Gamma_{D'}^S \in \mathsf{lfp}(W_\Pi^S)$ by induction hypothesis.

Suppose we have shown the claim for q, i.e. for all $D' \in I_q$, respectively $\neg D' \in I_q$, $q \leq r$, that $\Gamma_{D'}^S \in \mathsf{lfp}(W_\Pi^S)$ and consider I_{q-1}.
Let $\neg D' \in I_{q-1}$. No rule is applicable to $\neg D'$ and it occurs unchanged in I_q, thus $\Gamma_{D'}^S \in \mathsf{lfp}(W_\Pi^S)$.
Let $D' \in I_{q-1}$. If D' is not used for the derivation $I_{q-1} \Leftarrow I_q$ then it occurs unchanged in I_q, thus $\Gamma_{D'}^S \in \mathsf{lfp}(W_\Pi^S)$ as well. So let D' be the disjunction which is used for the derivation $I_{q-1} \Leftarrow I_q$. Analogous to the proof of Lemma 3.4, we can construct an active strong derivation sequence $\{D'\}, I'_1, \ldots, I'_{r'}$ where $C' = \bar{I'_{r'}}$ is a subdisjunction of C and thus, by Lemma 3.3, $\neg C' \in M_{WF}^S$ and $s(C') \leq \beta$. Then by induction hypothesis $\Gamma_{C'}^S \in \mathsf{lfp}(W_\Pi^S)$. All elements of I'_1 also occur in I_q, so by assumption for all $D'' \in I'_1$, respectively $\neg D'' \in I'_1$, $\Gamma_{D''}^S \in \mathsf{lfp}(W_\Pi^S)$. Additionally, by Lemma 3.4, if $D'' \in I'_1$ then $D'' \in M_{WF}^S$ and if $\neg D'' \in I'_1$ then $\neg D'' \in M_{WF}^S$. But then, by definition of T_Π^S, we have that $\Gamma_{D'}^S \in \mathsf{lfp}(W_\Pi^S)$. ■

We now lift Lemma 3.3 to the operator W_Π^S.

Lemma 3.5. *Let Π be a disjunctive logic program and $\Gamma_D^S \in (W_\Pi^S \uparrow \alpha)$.*

1. *If $D \in M_{WF}^S$ then $\Gamma_{D'}^S \in (W_\Pi^S \uparrow \alpha)$ for all $D \subseteq D'$.*
2. *If $\neg D \in M_{WF}^S$ then $\Gamma_{D'}^S \in (W_\Pi^S \uparrow \alpha)$ for all $D' \subseteq D$.*

Proof. We proof the statements by transfinite induction on α.
Let $\alpha = 0$. $W_\Pi^S \uparrow 0 = \emptyset$, by Definition 3.3, and the claim holds automatically. Suppose the two properties hold for all ordinals β with $\beta < \alpha$ and that $\Gamma_D^S \in (W_\Pi^S \uparrow \alpha)$.

Let α be a successor ordinal, i.e. $\alpha = \beta + 1$. We then have that $\Gamma_D^S \in W_\Pi^S(W_\Pi^S \uparrow \beta)$ and $\Gamma_D^S \in (T_\Pi^S(W_\Pi^S \uparrow \beta) \cup U_\Pi^S(W_\Pi^S \uparrow \beta))$ by Definition 3.3 and consider two cases.
(1) If $D \in M_{WF}^S$ then Γ_D^S has a false child and thus $\Gamma_D^S \in T_\Pi^S(W_\Pi^S \uparrow \beta)$. Then Γ_D^S contains an active strong derivation sequence $\{D\}, I_1, \ldots, I_r$ with child $C = \bar{I_r}$, $I_1 = \{D_1, \ldots, D_n, \neg D_1, \ldots, \neg D_m\}$ where $\neg C \in M_{WF}^S(\Pi)$, $\Gamma_C^S \in (W_\Pi^S \uparrow \beta)$ if $C \neq \{\}$, $\Gamma_{D_i}^S \in (W_\Pi^S \uparrow \beta)$, $D_i \in M_{WF}^S$, $\Gamma_{D_j}^S \in (W_\Pi^S \uparrow \beta)$, $\neg D_j \in M_{WF}^S$, for

all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Consider any disjunction D' with $D \subseteq D'$. By Lemma 3.2, $\Gamma^S_{D'}$ also contains an active strong derivation sequence with child C' such that $C' \subseteq C$. Since $\neg C \in M^S_{WF}$, by Lemma 3.3, $\neg C' \in M^S_{WF}$. Furthermore, by induction hypothesis, we have that $\Gamma_{C'} \in (W^S_\Pi \uparrow \beta)$. Let $\{D'\} \Leftarrow I'_1$ be the first derivation of that active strong derivation sequence where $I'_1 = \{D'_1, \ldots, D'_{n'}, \neg D'_1, \ldots, \neg D'_{m'}\}$. We reconsider the way we obtained the corresponding child C' in the proof of Lemma 3.2 and have to consider two cases:

(a) If (S1) is applied for $\{D\} \Leftarrow I_1$ then $H \subseteq D$. Since $D \subseteq D'$, we also have $H \subseteq D'$ and (S1) is applied for $\{D'\} \Leftarrow I'_1$. Then, for each $D'_{i'} \in I'_1$, there is a $D_i \in I_1$ such that $D_i \subseteq D'_{i'}$ where $1 \leq i \leq n$ and $1 \leq i' \leq n'$. Since all D_i are true in M^S_{WF}, we know by Lemma 3.3 that each $D'_{i'} \in M^S_{WF}$. Moreover, by induction hypothesis, we have that $\Gamma_{D'_{i'}} \in (W^S_\Pi \uparrow \beta)$. The sets of negated atoms obtained in I_1 and I'_1 are identical, i.e. $\neg D'_{j'} \in M^S_{WF}$ and $\Gamma^S_{D'_{j'}} \in (W^S_\Pi \uparrow \beta)$ for all $j' = 1, \ldots, m'$. But then $\Gamma^S_{D'} \in T^S_\Pi(W^S_\Pi \uparrow \beta)$.

(b) If (S2) is applied for $\{D\} \Leftarrow I_1$ then $H \not\subseteq D$, $H \cap D \neq \emptyset$, $C = H \setminus D$. Since $D \subseteq D'$, we have $H \cap D' \neq \emptyset$ as well and either (S1) or (S2) is applied in the derivation step $\{D'\} \Leftarrow I'_1$. In both cases, for each $D'_{i'} \in I'_1$, there is a $D_i \in I_1$ such that $D_i \subseteq D'_{i'}$ where $1 \leq i \leq n$ and $1 \leq i' \leq n'$. Again, since all D_i are true in M^S_{WF}, we know by Lemma 3.3 that each $D'_{i'} \in M^S_{WF}$ and, by induction hypothesis, that $\Gamma_{D'_{i'}} \in (W^S_\Pi \uparrow \beta)$. If we applied (S1) for $\{D'\} \Leftarrow I'_1$ then for each $\neg D'_{j'} \in I'_1$ there is a $\neg D_j \in I_1$ such that $D_j = D'_{j'}$ where $1 \leq j \leq m$ and $1 \leq j' \leq m'$. In case of (S2) we have $H \setminus D' \subseteq H \setminus D$, so for each $\neg D'_{j'} \in I'_1$ there is a $\neg D_j \in I_1$ with $D_j = D'_{j'}$, $1 \leq j \leq m$ and $1 \leq j' \leq m'$. Thus in both cases, for all $j' = 1, \ldots, m'$, $\neg D'_{j'} \in M^S_{WF}$ and $\Gamma^S_{D'_{j'}} \in (W^S_\Pi \uparrow \beta)$ and thus $\Gamma^S_{D'} \in T^S_\Pi(W^S_\Pi \uparrow \beta)$ as well. Altogether, $\Gamma^S_{D'} \in W^S_\Pi(W^S_\Pi \uparrow \beta)$, no matter whether (S1) or (S2) is applied in the first derivation step and thus $\Gamma^S_{D'} \in W^S_\Pi \uparrow \alpha$.

(2) If $\neg D \in M^S_{WF}$ then all children C in Γ^S_D are true and $\Gamma^S_D \in U^S_\Pi(W^S_\Pi \uparrow \beta)$. Then for each active strong derivation sequence in Γ^S_D with true child C we have $\Gamma^S_C \in (W^S_\Pi \uparrow \beta)$. Consider any D' with $D' \subseteq D$. If $\Gamma^S_{D'}$ does not contain any active derivation sequences then $\Gamma^S_{D'} \in (W^S_\Pi \uparrow 1)$ and the claim holds by monotonicity. Thus consider alternatively any active strong derivation sequence with child C'. Then, by Lemma 3.2, Γ^S_D also contains an active strong derivation sequence with child C such that $C \subseteq C'$. Since C is true in M^S_{WF}, by Lemma 3.3, $C' \in M^S_{WF}$ as well. Furthermore, by induction hypothesis, we have that $\Gamma_{C'} \in (W^S_\Pi \uparrow \beta)$. Since this holds for each child in $\Gamma^S_{D'}$ we know by definition of U^S_Π that $\Gamma^S_{D'} \in U^S_\Pi(W^S_\Pi \uparrow \beta)$ and thus $\Gamma^S_{D'} \in W^S_\Pi(W^S_\Pi \uparrow \beta)$, hence $\Gamma^S_{D'} \in W^S_\Pi \uparrow \alpha$.

Alternatively, let α be a limit ordinal. Then $W^S_\Pi \uparrow \alpha = \bigcup_{\beta<\alpha} W^S_\Pi \uparrow \beta$. If $\Gamma^S_D \in \bigcup_{\beta<\alpha}(W^S_\Pi \uparrow \beta)$ then there is a least ordinal β such that $\Gamma^S_D \in (W^S_\Pi \uparrow \beta)$ with $\alpha > \beta$. Assume that β is itself a limit ordinal. Then $\Gamma^S_D \in \bigcup_{\gamma<\beta} W^S_\Pi \uparrow \gamma$ and β is not the least ordinal such that $\Gamma^S_D \in W^S_\Pi \uparrow \beta$. So β has to be a successor ordinal. In this case we can apply the very same argument we used in case that α is a successor ordinal since $\beta < \alpha$. ∎

In the following, we present the alternative level mapping characterization.

Definition 3.4. *Let Π be a disjunctive logic program, let I be a model for Π, and let l be a disjunctive partial level mapping for Π. We say that Π satisfies (SWF) with respect to I and l if each $D \in \mathsf{dom}(l)$ satisfies one of the following conditions:*

(SWFi) $D \in I$ and Γ_D^S contains an active strong derivation sequence with child C, $\neg C \in I$ and $l(D) > l(C)$ if $C \neq \{\}$, and there is a clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ which is used for the first derivation of that sequence such that $\neg B_j \in I$ and $l(D) > l(B_j)$, $1 \leq j \leq m$, and one of the following conditions holds:

(SWFia) $H \subseteq D$ such that there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$ and $l(D) > l(D_i \vee A_i)$, $1 \leq i \leq n$.

(SWFib) $H \not\subseteq D$, $H \cap D \neq \emptyset$, $\{C_1, \ldots, C_l\} = H \setminus D$, $A_i \in I$ and $l(D) > l(A_i)$, $1 \leq i \leq n$, and $\neg C_k \in I$ and $l(D) > l(C_k)$, $1 \leq k \leq l$.

(SWFii) $\neg D \in I$ and for each active strong derivation sequence in Γ_D^S with child $C \in I$ there is a clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ which is used for the first derivation of that sequence such that (at least) one of the following conditions holds:

(SWFiia') $H \subseteq D$ and there exists i, $1 \leq i \leq n$, with $\neg(A_i \vee D) \in I$, $l(D) \geq l(A_i \vee D)$.

(SWFiia") $H \not\subseteq D$, $H \cap D \neq \emptyset$, and there exists i with $\neg A_i \in I$, $l(D) \geq l(A_i)$, $1 \leq i \leq n$.

(SWFiib') $H \subseteq D$ and there exists D' with $D' \subseteq B$, $D' \in I$ and $l(D) > l(D')$.

(SWFiib") $H \not\subseteq D$, $H \cap D \neq \emptyset$, $C = (H \setminus D)$, and there exists D' with $D' \subseteq (B \cup C)$, $D' \in I$ and $l(D) > l(D')$.

(SWFiic) $l(D) > l(C)$.

Theorem 3.1. *Let Π be a disjunctive program with strong well-founded model M. Then, in the disjunctive knowledge ordering, M is the greatest model amongst all models I for which there exists a disjunctive I-partial level mapping l for Π such that Π satisfies (SWF) with respect to I and l.*

Proof. Let M be the strong well-founded model of Π. We define the disjunctive M-partial level mapping l in the following way: $l(D) = \alpha$, where α is the least ordinal such that $\Gamma_D^S \in (W_\Pi^S \uparrow (\alpha+1)) = W_\Pi^S(W_\Pi^S \uparrow \alpha)$. This mapping is well defined since we have shown in Lemma 3.2 that the tree of each disjunctive literal which occurs in M is contained in the least fixed point of W_Π^S. We show that Π satisfies (SWF) with respect to M and l. Let $D \in \mathsf{dom}(l)$ and $l(D) = \alpha$. We have to consider two cases:
(1) If $D \in M$ then there is an active strong derivation sequence in Γ_D^S with false child. Since $\Gamma_D^S \in W_\Pi^S(W_\Pi^S \uparrow \alpha) = T_\Pi^S(W_\Pi^S \uparrow \alpha) \cup U_\Pi^S(W_\Pi^S \uparrow \alpha)$, we know that $\Gamma_D^S \in T_\Pi^S(W_\Pi^S \uparrow \alpha)$ by definition of T_Π^S and U_Π^S. By Definition 3.3, Γ_D^S contains an active strong derivation sequence $\{D\}, I_1, \ldots, I_r$ with child $C = \bar{I}_r$ and $I_1 = \{D_1, \ldots, D_n, \neg D_1, \ldots, \neg D_m\}$ where $\neg C \in M$, $\Gamma_C^S \in (W_\Pi^S \uparrow \alpha)$ if $C \neq$

$\{\}$, $\Gamma^S_{D_i} \in (W^S_\Pi \uparrow \alpha)$, $D_i \in M$, $\Gamma^S_{D_j} \in (W^S_\Pi \uparrow \alpha)$, $\neg D_j \in M$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Let $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_{m'}$ be the clause used in the first derivation step $\{D\} \Leftarrow I_1$. If we use (S1) for that derivation step then $H \subseteq D$ and $I_1 = \{A_1 \vee D, \ldots, A_n \vee D, \neg B_1, \ldots, \neg B_{m'}\}$. We already know that all positive disjunctions and all negative literals occurring in I_1 are contained in M. Moreover, since all these disjunctions, negative literals and the child C occur in $(W^S_\Pi \uparrow \alpha)$ we know that $l(A_i \vee D) < \alpha$, $l(B_j) < \alpha$, and $l(C) < \alpha$ if $C \neq \{\}$. Then D satisfies (SWFia) with respect to M and l where $D_i = D$ for each D_i in (SWFia). If we use (S2) for the first derivation step $\{D\} \Leftarrow I_1$ then $H \not\subseteq D$, $\{C_1, \ldots, C_l\} = H \setminus D$, and $I_1 = \{A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_{m'}, \neg C_1, \ldots, \neg C_l\}$. Again, all the positive disjunctions and all negative literals occurring in I_1 are contained in M. Likewise, all these disjunctions, negative literals, and the child C occur in $(W^S_\Pi \uparrow \alpha)$ and $l(A_i) < \alpha$, $l(B_j) < \alpha$, $l(C_k) < \alpha$ and $l(C) < \alpha$. Then D satisfies (SWFib) with respect to M and l.

(2) If $\neg D \in M$ then each child C of an active strong derivation sequence in Γ^S_D is true, thus $\Gamma^S_D \in U^S_\Pi(W^S_\Pi \uparrow \alpha)$ since $\Gamma^S_D \in W^S_\Pi(W^S_\Pi \uparrow \alpha) = T^S_\Pi(W^S_\Pi \uparrow \alpha) \cup U^S_\Pi(W^S_\Pi \uparrow \alpha)$. Then by Definition 3.3, for all active strong derivation sequences in Γ^S_D the corresponding child C is true in M and $\Gamma^S_C \in (W^S_\Pi \uparrow \alpha)$. Then $l_S(C) < \alpha$ and, for all derivation sequences with child C, D satisfies (SWFiic).

Alternatively, we show that if I is a model of Π and l a disjunctive I-partial level mapping such that Π satisfies (SWF) with respect to I and l then $I \subseteq M^S_{WF}$. We show via transfinite induction on $\alpha = l(D)$, that whenever $D \in I$, respectively $\neg D \in I$, then $\Gamma^S_D \in (W^S_\Pi \uparrow (\alpha + 1))$, i.e. $\Gamma^S_D \in W^S_\Pi(W^S_\Pi \uparrow \alpha) = T^S_\Pi(W^S_\Pi \uparrow \alpha) \cup U^S_\Pi(W^S_\Pi \uparrow \alpha)$ and thus $\Gamma^S_D \in \mathsf{lfp}(W^S_\Pi)$. Then D, respectively $\neg D$, occurs in M^S_{WF}.

Let $l(D) = 0$. If $D \in I$ then by (SWFi) Γ^S_D contains an active strong derivation sequence with child C, $\neg C \in I$ and $l(D) > l(C)$ if $C \neq \{\}$, and there is a clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ which is used for the first derivation of that sequence such that $\neg B_j \in I$ and $l(D) > l(B_j)$, $1 \leq j \leq m$, and either (SWFia) or (SWFib) holds. If (SWFia) holds then $H \subseteq D$ such that there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$ and $l(D) > l(D_i \vee A_i)$, $1 \leq i \leq n$. But there is no ordinal smaller than 0, i.e. the clause is a fact. Thus there is an active strong derivation sequence $\{D\} \Leftarrow \{\}$ with false child and $\Gamma^S_D \in (W^S_\Pi \uparrow 1)$ by definition of T^S_Π. If (SWFib) holds then $H \not\subseteq D$, $H \cap D \neq \emptyset$, $\{C_1, \ldots, C_l\} = H \setminus D$, $A_i \in I$ and $l(D) > l(A_i)$, $1 \leq i \leq n$, and $\neg C_k \in I$ and $l(D) > l(C_k)$, $1 \leq k \leq l$. There is no ordinal smaller than 0, so there can be no C_k in I_1 and (SWFib) cannot hold.

If $\neg D \in I$ then by (SWFii) for each active strong derivation in Γ^S_D with child $C \in I$ there is a clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ which is used for the first derivation of that sequence such that at least one of (SWFiia'), (SWFiia"), (SWFiib'), (SWFiib"), and (SWFiic) holds. If (SWFiia') holds then $H \subseteq D$ and there exists i, $1 \leq i \leq n$, with $\neg(A_i \vee D) \in I$, $l(D) \geq l(A_i \vee D)$. If (SWFiia") is satisfied then $H \not\subseteq D$, $H \cap D \neq \emptyset$, and there exists i with $\neg A_i \in I$, $l(D) \geq l(A_i)$, $1 \leq i \leq n$. If (SWFiib') holds then $H \subseteq D$ and there exists D' with $D' \subseteq B$, $D' \in I$ and $l(D) > l(D')$. If (SWFiib") is satisfied then $H \not\subseteq D$,

$H \cap D \neq \emptyset$, $C = (H \setminus D)$, and there exists D' with $D' \subseteq (B \cup C)$, $D' \in I$ and $l(D) > l(D')$ and if (SWFiic) holds then $l(D) > l(C)$. Since there is no ordinal smaller than 0, (SWFiib'), (SWFiib"), and (SWFiic) cannot hold. For the same reason, in case of (SWFiia') and (SWFiia") we can only have $l(D) = l(A_i \vee D)$, $l(D) = l(A_i)$ respectively. Since $(A_i \vee D)$, respectively A_i, is false in I, it has to satisfy (SWF) and the only possibility is again (SWFiia') or (SWFiia") with a positive disjunction of the same level which is false in I. Then this disjunction also has to satisfy (SWF) and the argument can be applied infinitely often. But the considered derivation sequence is active, thus we know that it is finite, so neither (SWFiia') nor (SWFiia") can hold. Hence, Γ_D^S does not have any children and thus $\Gamma_D^S \in (U_\Pi^S \uparrow 1)$ and $\Gamma_D^S \in (W_\Pi^S \uparrow 1)$.

Assume for all $D' \in EB_\Pi$ with $l(D') < \alpha$ that if $D' \in I$, respectively $\neg D' \in I$, then $\Gamma_{D'}^S \in W_\Pi^S \uparrow \alpha$ and let $l(D) = \alpha$. We have to consider two cases again.

If $D \in I$ then, by (SWFi), Γ_D^S contains an active strong derivation sequence with child C, $\neg C \in I$ and $l(D) > l(C)$ if $C \neq \emptyset$, and there is a clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ which is used for the first derivation of that sequence such that $\neg B_j \in I$ and $l(D) > l(B_j)$, $1 \leq j \leq m$, and either (SWFia) or (SWFib) holds. If (SWFia) holds then $H \subseteq D$ such that there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$ and $l(D) > l(D_i \vee A_i)$, $1 \leq i \leq n$. By Lemma 3.3, we know that also $(A_i \vee D) \in I$ and $l(A_i \vee D) \leq l(D_i \vee A_i)$ for all i. Then $I_1 = \{A_1 \vee D, \ldots, A_n \vee D, \neg B_1, \ldots, \neg B_m\}$ is the derivate of the first derivation of that sequence. By induction hypothesis, the trees of all elements contained in I_1 and Γ_C^S occur in $W_\Pi^S \uparrow \alpha$ and thus in the least fixed point of W_Π^S. Then all elements of I_1 and $\neg C$ are contained in M_{WF}^S. Hence $\Gamma_D^S \in T_\Pi^S \uparrow (\alpha + 1)$, by Definition 3.3, and $\Gamma_D^S \in W_\Pi^S \uparrow (\alpha + 1)$. If (SWFib) holds then $H \not\subseteq D$, $H \cap D \neq \emptyset$, $\{C_1, \ldots, C_l\} = H \setminus D$, $A_i \in I$, $A_i \in I$ and $l(D) > l(A_i)$, $1 \leq i \leq n$, and $\neg C_k \in I$ and $l(D) > l(C_k)$, $1 \leq k \leq l$. Then $I_1 = \{A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m, \neg C_1, \ldots, \neg C_l\}$ is the derivate of the first derivation of that sequence. By induction hypothesis, the trees of all elements contained in I_1 and Γ_C^S occur in $W_\Pi^S \uparrow \alpha$ and thus, again, all elements of I_1 and $\neg C$ are contained in M_{WF}^S. Hence $\Gamma_D^S \in T_\Pi^S \uparrow (\alpha + 1)$, by Definition 3.3, and thus $\Gamma_D^S \in W_\Pi^S \uparrow (\alpha + 1)$.

If $\neg D \in I$ then, by (SWFii), for each active strong derivation in Γ_D^S with child $C \in I$ there is a clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ which is used for the first derivation of that sequence such that at least one of (SWFiia'), (SWFiia"), (SWFiib'), (SWFiib"), and (SWFiic) holds. Consider such an arbitrary active strong derivation. We show that $l(D) > l(C)$ holds in all these cases.

- If (SWFiic) holds then $l(D) > l(C)$ is satisfied automatically.
- If (SWFiib') holds then $H \subseteq D$ and there exists D' with $D' \subseteq B$, $D' \in I$ and $l(D) > l(D')$. We have that all $\neg B_j$, $j = 1 \ldots, m$, occur in I_1, the first derivate of that sequence. No more rule is applicable to any $\neg B_j$ and it also occurs in the basis, thus $B_j \in C$ and so $D' \subseteq C$. By induction hypothesis,

$\mathit{\Gamma}^S_{D'} \in (W^S_\Pi \uparrow \alpha)$. Then, by Lemma 3.5, $\mathit{\Gamma}_C \in (W^S_\Pi \uparrow \alpha)$ as well and $l(C) < \alpha$ and thus $l(D) > l(C)$.

- If (SWFiib") is satisfied then $H \not\subseteq D$, $H \cap D \neq \emptyset$, $C' = (H \setminus D)$, and there exists D' with $D' \subseteq (B \cup C')$, $D' \in I$ and $l(D) > l(D')$. All elements in $B \cup C'$ occur negated in I_1, the first derivate of that sequence. No more rule is applicable to any negated atom and it also occurs in the basis, thus $D' \subseteq C$. By induction hypothesis, $\mathit{\Gamma}^S_{D'} \in (W^S_\Pi \uparrow \alpha)$. Then, by Lemma 3.5, $\mathit{\Gamma}_C \in (W^S_\Pi \uparrow \alpha)$ as well and $l(C) < \alpha$ and thus $l(D) > l(C)$.
- If (SWFiia') holds then $H \subseteq D$ and there exists i, $1 \leq i \leq n$, with $\neg(A_i \vee D) \in I$, $l(D) \geq l(A_i \vee D)$ and if (SWFiia") is satisfied then $H \not\subseteq D$, $H \cap D \neq \emptyset$, and there exists i with $\neg A_i \in I$, $l(D) \geq l(A_i)$, $1 \leq i \leq n$. We join these two cases since the argument is exactly the same and prove for all positive disjunctions D' occurring in the active strong derivation sequence $\{D\}, I_1, \ldots, I_r$ with $\neg D' \in M^S_{WF}$, $l(D') \leq l(D)$, and D' satisfies (SWFiia') or (SWFiia") that $l(C) < l(D)$. Then the claim also holds for D itself.
  Since I_r is the basis, D' cannot occur in I_r. The same holds for I_{r-1} because I_r cannot contain any $A_i \vee D'$ or A_i which would be necessary for $D' \in I_{r-1}$ satisfying (SWFiia') or (SWFiia"). So let $D' \in I_{r-2}$. Then I_{r-1} contains (at least) one disjunction D'' with $D'' = A_i \vee D'$ or $D'' = A_i$ but in both cases D'' cannot satisfy (SWFiia') or (SWFiia") as already mentioned. Since D'' occurs in I_{r-1}, the clause which is used for $I_{r-1} \Leftarrow I_r$ cannot contain positive literals in the body because I_r is the basis. Thus there is also an active strong derivation sequence in $\mathit{\Gamma}^S_{D''}$ using this clause with child C'. Since $\neg D'' \in I$ we know that D'' satisfies (SWFii) and it cannot satisfy (SWFiia') or (SWFiia") so $l(C') < l(D'')$ as we have already shown for the other cases. We know $l(D) \geq l(D')$ and $l(D') \geq l(D'')$ so we can apply the induction hypothesis and $C' \in (W^S_\Pi \uparrow \alpha)$ and thus $C' \in M^S_{WF}$. By construction, C' is a subset of the child C for the considered sequence in $\mathit{\Gamma}^S_D$ and, by Lemma 3.5, we know that $C \in (W^S_\Pi \uparrow \alpha)$. Thus $l(C) < l(D)$.
  Finally, consider for all positive disjunctions occurring in I_q with $\neg D' \in M^S_{WF}$, $l(D') \leq l(D)$, and D' satisfies (SWFiia') or (SWFiia") that $l(C) < l(D)$ has been shown. We show that it also holds for all these D' in I_{q-1}. If D' is not used in the derivation step $I_{q-1} \Leftarrow I_q$ then it occurs unchanged in I_q and the claim holds by assumption. So let D' be the disjunction used for that derivation step such that there is either $A_i \vee D'$ or A_i with $l(D') \geq l(A_i \vee D')$, respectively $l(D') \geq l(A_i)$. If $A_i \vee D$, respectively A_i, satisfies (SWFiia') or (SWFiia") then the claim has already been proven. If not, then we can apply the very same argument for I_{q-1} which we used for I_{r-2} above, and the claim holds as well.

Thus, for all derivation sequences with child C, we have $l(D) > l(C)$ and $\mathit{\Gamma}^S_C \in (W^S_\Pi \uparrow \alpha)$. Then $\mathit{\Gamma}^S_D \in U^S_\Pi(W^S_\Pi \uparrow \alpha)$ by definition of U_Π and $\mathit{\Gamma}^S_D \in W^S_\Pi \uparrow (\alpha + 1)$ by Definition 3.3. ■

The characterization contains obviously much more complicated conditions than Definition 2.1. We postpone a comparison and continue instead the example.

Example 3.6. *(Example 3.1 continued)* Considering the assignments of the levels, we have $l(l \vee r) = 0$ by (SWFia) and $l(e) = 1$ by (SWFiia') and therefore $l(f) = 2$ by (SWFia). Moreover, $l(b) = 1$ by (SWFiib') whereas $l(c) = 1$ by (SWFiib").

Once more we mention that without the problems shown in Table 1 the condition (SWFi) would not contain the case (SWFib) and even the reference to the active derivation sequence and the child would not appear. It is only necessary for proving the statement which we cannot do in a better way without solving the mentioned problems.

In case of (SWFii) we only used (SWFiic) for the proof, respectively reduced all the other cases to (SWFiic). Thus the following corollary is straightforward.

Corollary 3.1. *Let Π be a disjunctive program with strong well-founded model M. Then, in the disjunctive knowledge ordering, M is the greatest model amongst all models I for which there exists a disjunctive I-partial level mapping l for Π such that Π satisfies (SWF') with respect to I and l where (SWF') is (SWF) substituting (SWFii) by (SWFii').*

> *(SWFii') $\neg D \in I$ and for each active strong derivation in Γ_D^S with child $C \in I$ we have $l(D) > l(C)$.*

This is apparently much shorter then Definition 3.4. We just prefer the more detailed version because it allows to compare the result in a better way to other characterizations.
Moreover, (SWFiic) does not only cover the four other conditions but also combines knowledge derived from several clauses.

Example 3.7. Let Π be the following program.

$$\begin{aligned} p \vee q &\leftarrow r \\ r &\leftarrow \neg s \\ q \vee s &\leftarrow \end{aligned}$$

We have $\{p\} \Leftarrow \{r, \neg q\} \Leftarrow \{\neg q, \neg s\}$ as the only active derivation sequence in Γ_p^S. Since $q \vee s$, the child, is true, p has to be false. However neither $\neg q$, nor r, nor $\neg s$ are false in the strong well-founded model. Thus (SWF) only holds in this case because of (SWFiic).

4 Generalized Disjunctive Well-founded Semantics

Baral, Lobo, and Minker introduced GDWFS ([2]) based on state-pairs. They applied various operators for calculating the semantics and we recall at first $\mathcal{T}_S^D$ and $\mathcal{F}_S^D$ for disjunctive programs.

Definition 4.1. *Let S be a state-pair and Π be a disjunctive program. Let $T \subseteq EB_\Pi$ and $F \subseteq CB_\Pi$.*

$\mathcal{T}_S^D(T) = \{D \in EB_\Pi \mid D$ *undefined in* S, $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ *in* $\mathsf{ground}(\Pi)$ *such that for all* i, $1 \leq i \leq n$, $(A_i \vee D_i) \in S$ *or* $(A_i \vee D_i) \in T$, D_i *might be empty*, $\neg B_j \in S$ *for all* j, $1 \leq j \leq m$, *and* $(H \cup \bigcup_i D_i) \subseteq D.\}$

$\mathcal{F}_S^D(F) = \{C \in CB_\Pi \mid C$ *is undefined in* S, $A \in C$, *and for all clauses* $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ *in* $\mathsf{ground}(\Pi)$, *with* $A \in H$, *at least one of the following three cases holds:* $(B_1 \vee \cdots \vee B_m) \in S$, $\neg(A_1 \wedge \cdots \wedge A_n) \in S$, *or* $\neg(A_1 \wedge \cdots \wedge A_n) \in F\}$

T_S^D is bottom-up and F_S^D is top-down: $\mathcal{T}_S^D \uparrow 0 = \emptyset$, $\mathcal{T}_S^D \uparrow (n+1) = \mathcal{T}_S^D(\mathcal{T}_S^D \uparrow n)$, $\mathcal{T}_S^D = \bigcup_{n<\omega} \mathcal{T}_S^D \uparrow n$, and $\mathcal{F}_S^D \downarrow 0 = CB_\Pi$, $\mathcal{F}_S^D \downarrow (n+1) = \mathcal{F}_S^D(\mathcal{F}_S^D \downarrow n)$, $\mathcal{F}_S^D = \bigcap_{n<\omega} \mathcal{F}_S^D \downarrow n$.
There are two more operators defined for definite programs which necessitates the following program transformations. Given a disjunctive program Π and a state-pair S, $DIS(\Pi)$ is obtained by transferring all negated atoms in the body of each clause of Π as atoms to its head. Then, $Dis(\Pi, S)$ results from $DIS(\Pi)$ by reducing the clauses in $DIS(\Pi)$ as follows: remove atoms from the body of a clause if they are true in S, remove a clause if its head is true in S, and remove atoms from the head of a clause if they are false in S. This is similar to the construction used for stable models and we recall $T_\Pi^D(T)$, a simplification of $\mathcal{T}_S^D(T)$. Given a definite (disjunctive) program Π and T, a subset of EB_Π, we have that $T_\Pi^D(T) = \{D \in EB_\Pi \mid H \leftarrow A_1, \ldots, A_n$ in $\mathsf{ground}(\Pi)$ such that for all i, $1 \leq i \leq n$, $(A_i \vee D_i) \in T$, D_i might be empty, and $(H \cup \bigcup_i D_i) \subseteq D.\}$ We then iterate $T_\Pi^D \uparrow 0 = \emptyset$, $T_\Pi^D \uparrow (n+1) = T_\Pi^D(T_\Pi^D \uparrow n)$, $T_\Pi^D = \bigcup_{n<\omega} T_\Pi^D \uparrow n$.
For deriving indefinite false conjunctions the Extended Generalized Closed World Assumption (EGCWA) ([21]) is applied. It intuitively says that a conjunction can be inferred to be false from Π if and only if it is false in all minimal models of Π where a minimal model [12] is a two-valued model M of Π such that no subset of it is a model as well.
The previous two constructions yield the operators $T_S^{ED} = \{D \mid D \in T_{Dis(\Pi,S)}^D$ and $D \notin S\}$ and $F_S^{ED} = \{C \mid C \in \text{EGCWA}(Dis(\Pi, S) \cup S)$ and $C \notin S\}$. Now we can combine all the operators and obtain $\mathcal{S}^{ED}(S) = S \cup \mathcal{T}_S^D \cup \neg\mathcal{F}_S^D \cup T_S^{ED} \cup \neg F_S^{ED}$. The iteration is done via $M_0 = \emptyset$, $M_{\alpha+1} = \mathcal{S}^{ED}(M_\alpha)$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$, for limit ordinal α and has a fixed point ([2]). The fixed point corresponds to the generalized disjunctive well-founded model M_Π^{ED} which is consistent ([2]).

Example 4.1. Recall the program from Example 3.1. We have $M_\Pi^{ED} = \{l \vee r, q, f, \neg p, \neg b, \neg c, \neg e, \neg g, \neg(l \wedge r)\}$. Note that M_Π^{ED} is closed in so far that any superset of a true disjunction (false conjunction) is true (false) as well.

We continue with the level mapping characterization of GDWFS.

Definition 4.2. *Let* Π *be a disjunctive logic program, let the state-pair* I *be a model for* Π, *and let* l_1, l_2 *be disjunctive* I*-partial level mappings for* Π. *We say that* Π *satisfies (GDWF) with respect to* I, l_1, *and* l_2 *if each* $D \in \mathsf{dom}(l_1)$ *and each* $\neg C \in \mathsf{dom}(l_1)$ *satisfies one of the following conditions:*

(GDWFi) $D \in I$ *and there is a clause* $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ *in* $\mathsf{ground}(\Pi)$ *with* $H \subseteq D$ *such that* $\neg B_j \in I$ *and* $l_1(D) >_1 l_t(\neg B_j)$, $t \in \{1, 2\}$,

for all $j = 1, \ldots, m$ *and, for all* $i = 1, \ldots, n$, *there is* $D_i \subseteq D$ *with* $(D_i \vee A_i) \in I$ *where* $l_1(D) > l_1(D_i \vee A_i)$ *or* $l_1(D) >_1 l_2(D_i \vee A_i)$.
(GDWFii) $\neg C \in I$ *with atom* $A \in C$ *and for each clause* $H \leftarrow A_1, \ldots, A_n$, $\neg B_1, \ldots, \neg B_m$ *in* $\mathsf{ground}(\Pi)$ *with* $A \in H$ *(at least) one of the following conditions holds:*

(GDWFiia) $\neg(A_1 \wedge \ldots \wedge A_n) \in I$ *and* $l_1(\neg C) \geq l_1(\neg(A_1 \wedge \ldots \wedge A_n))$.
(GDWFiia') $\neg(A_1 \wedge \ldots \wedge A_n) \in I$ *and* $l_1(\neg C) >_1 l_2(\neg(A_1 \wedge \ldots \wedge A_n))$.
(GDWFiib) $(B_1 \vee \ldots \vee B_m) \in I$ *and* $l_1(\neg C) >_1 l_t(B_1 \vee \ldots \vee B_m)$ *for* $t \in \{1, 2\}$.

and each $D, \neg C \in \mathsf{dom}(l_2)$ *satisfies one of the following conditions:*

(GDWFi') $D \in I$ *and there is a clause* $H_1 \vee \cdots \vee H_l \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ *in* $\mathsf{ground}(\Pi)$ *such that* $\emptyset \neq ((H \cup B) \setminus D') \subseteq D$, $H_k \in D'$ *for each* $\neg H_k \in I$ *with* $l_2(D) >_1 l_t(\neg H_k)$, $t \in \{1, 2\}$, $B_j \in D'$ *for each* $\neg B_j \in I$ *with* $l_2(D) >_1 l_t(\neg B_j)$, $t \in \{1, 2\}$, *for all* $k = 1, \ldots, l$ *and all* $j = 1, \ldots, m$, *and, for all* $i = 1, \ldots, n$, *there is* $D_i \subseteq D$ *with* $(D_i \vee A_i) \in I$ *where* $l_2(D) >_2 l_2(D_i \vee A_i)$ *or* $A_i \in I$ *where* $l_2(D) >_1 l_s(A_i)$, $s \in \{1, 2\}$.
(GDWFii') $\neg C \in I$ *and* $C \in EGCWA(Dis(\Pi, S) \cup S)$, $C \notin S$ *and* $l_2(\neg C) >_1 l_t(L)$, $t \in \{1, 2\}$, *if and only if* $L \in S$.

The reason for introducing two mappings is to extrapolate exactly the simultaneous iteration of the two operators dealing with positive, negative respectively, information. The theorem stating the equivalence is given in the following.

Theorem 4.1. *Let* Π *be a disjunctive program with generalized disjunctive well-founded model* M. *Then, in the disjunctive knowledge ordering,* M *is the greatest model amongst all models* I *for which there exist disjunctive* I*-partial level mappings* l_1 *and* l_2 *for* Π *such that* Π *satisfies (GDWF) with respect to* I, l_1, *and* l_2.

Proof. Let M be the generalized disjunctive well-founded model of Π. We define the disjunctive M-partial level mappings l_1 and l_2 in the following way: If $D \in \mathcal{T}^D_{M_\alpha}$ then β is the least ordinal such that $D \in \mathcal{T}^D_{M_\alpha} \uparrow (\beta + 1)$ and $l_1(D) = (\alpha, \beta)$. If $D \in T^{ED}_{M_\alpha}$ then β is the least ordinal such that $D \in T^D_{Dis(\Pi, M_\alpha)} \uparrow (\beta + 1)$ and $l_2(D) = (\alpha, \beta)$. If $C \in \mathcal{F}^D_{M_\alpha}$ then $l_1(\neg C) = (\alpha, 0)$. If $C \in F^{ED}_{M_\alpha}$ then $l_2(\neg C) = (\alpha, 0)$. All other values remain undefined. By Definition of M, we know that any $D \in M$, respectively $\neg C \in M$, is at least contained in one of $\mathcal{T}^D_{M_\alpha}$, $\neg \mathcal{F}^D_{M_\alpha}$, $T^{ED}_{M_\alpha}$, and $\neg F^{ED}_{M_\alpha}$ for some α and thus in the domain of l_1 or l_2. We show that Π satisfies (GDWF) with respect to M, l_1, and l_2 and consider the following four cases:
(1) Let $D \in \mathsf{dom}(l_1)$ with $l_1(D) = (\alpha, \beta)$. By definition of l_1 we have that $D \in \mathcal{T}^D_{M_\alpha}$. We know that β is the least ordinal such that $D \in \mathcal{T}^D_{M_\alpha} \uparrow (\beta + 1)$ and thus $D \in \mathcal{T}^D_{M_\alpha}(\mathcal{T}^D_{M_\alpha} \uparrow \beta)$. Then, by Definition 4.1, D is undefined in M_α and $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ such that for all i, $1 \leq i \leq n$, $(A_i \vee D_i) \in M_\alpha$ or $(A_i \vee D_i) \in (\mathcal{T}^D_{M_\alpha} \uparrow \beta)$, D_i might be empty, $\neg B_j \in M_\alpha$ for all j, $1 \leq j \leq m$, and $(H \cup \bigcup_i D_i) \subseteq D$. If $(A_i \vee D_i) \in (\mathcal{T}^D_{M_\alpha} \uparrow \beta)$ then

$(A_i \vee D_i) \in \mathcal{T}^D_{M_\alpha}$. Thus $(A_i \vee D_i) \in M_{\alpha+1}$ and $l_1(A_i \vee D_i) = (\alpha, \beta')$ with $\beta' < \beta$. Then $(A_i \vee D_i) \in M$ and $l_1(D) > l_1(A_i \vee D_i)$. If $(A_i \vee D_i) \in M_\alpha$ then $(A_i \vee D_i) \in M$ and $l_1(D) >_1 l_t(A_i \vee D_i)$, $t \in \{1,2\}$ and, likewise, for all $\neg B_j \in M_\alpha$, we have $\neg B_j \in M$ and $l_1(D) >_1 l_t(\neg B_j)$, $t \in \{1,2\}$. Then D satisfies (GDWFi).
(2) Let $\neg C \in \mathsf{dom}(l_1)$ with $l_1(\neg C) = (\alpha, 0)$. By definition of l_1 we have that $C \in \mathcal{F}^D_{M_\alpha}$. We know that $\mathcal{F}^D_{M_\alpha} = \bigcap_{n<\omega} \mathcal{F}^D_{M_\alpha} \downarrow n$ and thus, for all n, $C \in \mathcal{F}^D_{M_\alpha} \downarrow n$. Consider any $n = n' + 1$. We have $\mathcal{F}^D_{M_\alpha} \downarrow (n'+1) = \mathcal{F}^D_{M_\alpha}(\mathcal{F}^D_{M_\alpha} \downarrow n')$. By Definition 4.1, C is undefined in M_α, $A \in C$, and for all clauses $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$, with $A \in H$, at least one of the following three cases holds:

- We have $(B_1 \vee \cdots \vee B_m) \in M_\alpha$. Then $(B_1 \vee \cdots \vee B_m) \in M$, and $l_1(\neg C) >_1 l_t(B_1 \vee \cdots \vee B_m)$, $t \in \{1,2\}$, and $\neg C$ satisfies (GDWFiib).
- Or $\neg(A_1 \wedge \cdots \wedge A_n) \in M_\alpha$ holds. Then $\neg(A_1 \wedge \cdots \wedge A_n) \in M$ and $l_1(\neg C) >_1 l_1(\neg(A_1 \wedge \cdots \wedge A_n))$ and $\neg C$ satisfies (GDWFiia) or $l_1(\neg C) >_1 l_2(\neg(A_1 \wedge \cdots \wedge A_n))$and $\neg C$ satisfies (GDWFiia').
- Otherwise $\neg(A_1 \wedge \cdots \wedge A_n) \in (\mathcal{F}^D_{M_\alpha} \downarrow n')$. Since $C \in (\mathcal{F}^D_{M_\alpha} \downarrow n')$ for all n', we know that $\neg(A_1 \wedge \cdots \wedge A_n) \in (\mathcal{F}^D_{M_\alpha} \downarrow n')$ for all n', otherwise there had to be another reason for C occurring in all iterations of $\mathcal{F}^D_{M_\alpha}$. Thus $\neg(A_1 \wedge \cdots \wedge A_n) \in \mathcal{F}^D_{M_\alpha}$ and $\neg(A_1 \wedge \cdots \wedge A_n) \in M_{\alpha+1}$ and $l_1(\neg(A_1 \wedge \cdots \wedge A_n)) = (\alpha, 0)$ Therefore $l_1(\neg C) = l_1(\neg(A_1 \wedge \cdots \wedge A_n))$ and $\neg(A_1 \wedge \cdots \wedge A_n) \in M$. Hence, C satisfies (GDWFiia), as well.

(3) Alternatively, let $D \in \mathsf{dom}(l_2)$ and $l_2(D) = (\alpha, \beta)$. By definition of l_2 we have that $D \in T^{ED}_{M_\alpha}$. We know that β is the least ordinal such that $D \in T^D_{Dis(\Pi, M_\alpha)} \uparrow (\beta + 1)$ and thus $D \in T^D_{Dis(\Pi,M_\alpha)}(T^D_{Dis(\Pi,M_\alpha)} \uparrow \beta)$. Then $H \leftarrow A_1, \ldots, A_n$ in $\mathsf{ground}(Dis(\Pi, M_\alpha))$ such that for all i, $1 \le i \le n$, $(A_i \vee D_i) \in T^D_{Dis(\Pi,M_\alpha)} \uparrow \beta$, D_i might be empty, and $(H \cup \bigcup_i D_i) \subseteq D$. If $(A_i \vee D_i) \in (T^D_{Dis(\Pi,M_\alpha)} \uparrow \beta)$ then $(A_i \vee D_i) \in T^D_{Dis(\Pi,M_\alpha)}$ and thus $(A_i \vee D_i) \in T^{ED}_{M_\alpha}$. Then $(A_i \vee D_i) \in M_{\alpha+1}$ and $l_2(A_i \vee D_i) = (\alpha, \beta')$ with $\beta' < \beta$, so $(A_i \vee D_i) \in M$ and $l_2(D) >_2 l_2(A_i \vee D_i)$. Furthermore, we also have a clause $H \vee D' \leftarrow A_1, \ldots, A_n, A_{n+1}, \ldots, A_r$ in $\mathsf{ground}(Dis(\Pi))$. By definition of $Dis(\Pi, M_\alpha)$, all A_q, $q = n+1, \ldots, r$, occur in M_α and thus $A_q \in M$ and $l_2(D) >_1 l_t(A_q)$, $t \in \{1,2\}$. Likewise, for all elements $E \in D'$ we have $\neg E \in M_\alpha$. Then $\neg E \in M$ and $l_2(D) >_1 l_t(\neg E)$, $t \in \{1,2\}$. We also have a clause $H' \leftarrow A_1, \ldots, A_n, A_{n+1}, \ldots, A_r, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ with $H' \cup \{B_1, \ldots, B_m\} = H \cup D'$. Then $((H' \cup B) \setminus D') \subseteq D$ and (GDWFi') is satisfied.
(4) Finally, let $C \in \mathsf{dom}(l_2)$ with $l_2(\neg C) = (\alpha, 0)$. By definition of l_2 we have that $C \in F^{ED}_{M_\alpha}$. Then $C \in \mathrm{EGCWA}(Dis(\Pi, M_\alpha) \cup M_\alpha)$ and $C \notin M_\alpha$. By definition of l_1 and l_2, for all $L \in M_\alpha$ we have $l_2(C) >_1 l_t(L)$, $t \in \{1,2\}$. Thus, for all L with $l_2(C) >_1 l_t(L)$, $t \in \{1,2\}$, we have $L \in M_\alpha$. Hence, C satisfies (GDWFii').

Conversely, we show that if I is a model of Π and l_1 and l_2 are disjunctive I-partial level mappings such that Π satisfies (GDWF) with respect to I, l_1, and

l_2 then $I \subseteq M_\Pi^{ED}$. We show via transfinite induction on α that whenever $D \in I$ with $l_1(D) = (\alpha, \beta)$ or $l_2(D) = (\alpha, \beta)$ then $D \in M_{\alpha+1}$ and whenever $\neg C \in I$ with $l_1(\neg C) = (\alpha, \beta)$ or $l_2(\neg C) = (\alpha, \beta)$ then $\neg C \in M_{\alpha+1}$. This suffices to show that $D \in M_\Pi^{ED}$ and $\neg C \in M_\Pi^{ED}$.

Let $\alpha = 0$. We consider four cases.

(1) If $D \in I$ and $D \in \mathsf{dom}(l_1)$ then D satisfies (GDWFi) and there is a clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ with $H \subseteq D$ such that $\neg B_j \in I$ and $l_1(D) >_1 l_t(\neg B_j)$, $t \in \{1, 2\}$, for all $j = 1, \ldots, m$, and, for all $i = 1, \ldots, n$, there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$ where $l_1(D) > l_1(D_i \vee A_i)$ or $l_1(D) >_1 l_2(D_i \vee A_i)$. Since there is no ordinal smaller than 0, we know that all conditions including $>_1$ cannot be satisfied. Thus (GDWFi) simplifies to that there is a clause $H \leftarrow A_1, \ldots, A_n$ in $\mathsf{ground}(\Pi)$ with $H \subseteq D$ such that there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$ and $l_1(D) > l_1(D_i \vee A_i)$ for all $i = 1, \ldots, n$. We show by induction on β that $D \in \mathcal{T}_{M_0}^D \uparrow (\beta+1)$, thus $D \in \mathcal{T}_{M_0}^D$ and $D \in M_1$. Let $\beta = 0$. Since there is no ordinal smaller than 0, the considered clause is a fact. Then $D \in (\mathcal{T}_{M_0}^D \uparrow 1)$. Suppose the property holds for all D with $\beta' < \beta$ and let $l(D) = (0, \beta)$. We know that D satisfies the simplified (GDWFi) and, by assumption, for all $(D_i \vee A_i) \in I$ with $l_1(D) > l_1(D_i \vee A_i)$ we have $(D_i \vee A_i) \in (\mathcal{T}_{M_0}^D \uparrow \beta)$. Then, by Definition 4.1, $D \in \mathcal{T}_{M_0}^D \uparrow (\beta + 1)$.

(2) If $\neg C \in I$ and $\neg C \in \mathsf{dom}(l_1)$ then C satisfies (GDWFii) and there is an atom $A \in D$ such that for each clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ with $A \in H$ (at least) one of the conditions (GDWFiia), (GDWFiia'), or (GDWFiib) holds. Since there is no ordinal smaller than 0, (GDWFiia') and (GDWFiib) cannot hold by definition of $>_1$. Thus (GDWFiia) holds for all clauses and there is $\neg(A_1 \wedge \ldots \wedge A_n) \in I$, $l_1(\neg D) \geq l_1(\neg(A_1 \wedge \ldots \wedge A_n))$, and for the same reason $l_1(\neg(A_1 \wedge \ldots \wedge A_n)) = (0, \beta')$ with $\beta' \leq \beta$. Thus $\neg(A_1 \wedge \ldots \wedge A_n)$ satisfies (GDWFiia) as well. We can apply the argument again and obtain eventually that each $\neg C$ which satisfies (GDWFiia) does this by means of a negated conjunction which satisfies also (GDWFiia). But then, for all clauses $H \leftarrow A_1, \ldots, A_n$, $\neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ with $A \in H$, $\neg(A_1 \wedge \cdots \wedge A_n) \in \mathcal{F}_{M_0}^D \downarrow n'$ for all n' and thus $C \in \bigcap_{n' < \omega} \mathcal{F}_{M_0}^D \downarrow n'$ and $C \in \mathcal{F}_{M_0}^D$. Hence, $\neg C \in M_1$.

(3) If $D \in I$ and $D \in \mathsf{dom}(l_2)$ then D satisfies (GDWFi') and there is a clause $H_1 \vee \cdots \vee H_l \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ such that $\emptyset \neq ((H \cup B) \setminus D') \subseteq D$, $H_k \in D'$ for each $\neg H_k \in I$ with $l_2(D) >_1 l_t(\neg H_k)$, $t \in \{1, 2\}$, $B_j \in D'$ for each $\neg B_j \in I$ with $l_2(D) >_1 l_t(\neg B_j)$, $t \in \{1, 2\}$, for all $k = 1, \ldots, l$ and all $j = 1, \ldots, m$, and, for all $i = 1, \ldots, n$, there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$ where $l_2(D) >_2 l_2(D_i \vee A_i)$ or $A_i \in I$ where $l_2(D) >_1 l_s(A_i)$, $s \in \{1, 2\}$. Since there is no ordinal smaller than 0 we know that all conditions including $>_1$ cannot be satisfied. Thus (GDWFi') simplifies to that there is a clause $H_1 \vee \cdots \vee H_l \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ such that there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$ and $l_2(D) >_2 l_2(D_i \vee A_i)$ for all $i = 1, \ldots, n$, $(H \cup B) \subseteq D$. Note in particular that the set D' has to be empty and that $((H \cup B) \setminus D') \neq \emptyset$ holds automatically. Then there is also a clause $H' \leftarrow A_1, \ldots, A_n$ with $H' = H \cup B$ in $DIS(\Pi)$ and, since M_0 is empty, also in $Dis(\Pi, M_0)$. We show by induction on β that $D \in T_{Dis(\Pi, M_0)}^D \uparrow (\beta + 1)$, thus $D \in T_{M_0}^{ED}$ and therefore $D \in M_1$.

Let $\beta = 0$. Since there is no ordinal smaller than 0, the considered clause is a fact and $D \in (T^D_{Dis(\Pi,M_0)} \uparrow 1)$. Suppose the property holds for all D with $\beta' < \beta$ and let $l(D) = (0, \beta)$. We know that D satisfies the simplified (GDWFi') and by assumption for all $(D_i \vee A_i) \in I$ with $l_2(D) >_2 l_2(D_i \vee A_i)$ we have $(D_i \vee A_i) \in (T^D_{Dis(\Pi,M_0)} \uparrow \beta)$. Then $D \in T^D_{Dis(\Pi,M_0)} \uparrow (\beta + 1)$.
(4) If $\neg C \in I$ and $\neg C \in \mathsf{dom}(l_2)$ then C satisfies (GDWFii') so that we have $C \in \text{EGCWA}(Dis(\Pi, S) \cup S)$, $C \notin S$ and $l_2(\neg C) >_1 l_t(L)$, $t \in \{1, 2\}$, if and only if $L \in S$. Since there is no ordinal smaller than 0, we know that S is empty. Then $C \in \text{EGCWA}(Dis(\Pi, M_0) \cup M_0)$, $C \notin M_0$ and $C \in F^{ED}_{M_0}$, i.e. $\neg C \in M_1$.

So suppose for all $\alpha' < \alpha$ that if $D \in I$ with $l_1(D) = (\alpha', \beta)$ or $l_2(D) = (\alpha', \beta)$ then $D \in M_{\alpha'+1}$ and if $\neg C \in I$ with $l_1(\neg C) = (\alpha', \beta)$ or $l_2(\neg C) = (\alpha', \beta)$ then $\neg C \in M_{\alpha'+1}$. We show that the property also holds for all $D \in I$ with $l_1(D) = (\alpha, \beta)$ or $l_2(D) = (\alpha, \beta)$ and all $\neg C \in I$ with $l_1(\neg C) = (\alpha, \beta)$ or $l_2(\neg C) = (\alpha, \beta)$ and consider again four cases.
(1) If $D \in I$ and $D \in \mathsf{dom}(l_1)$ then D satisfies (GDWFi) and there is a clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ with $H \subseteq D$ such that $\neg B_j \in I$ and $l_1(D) >_1 l_t(\neg B_j)$, $t \in \{1, 2\}$, for all $j = 1, \ldots, m$, and, for all $i = 1, \ldots, n$, there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$ where $l_1(D) > l_1(D_i \vee A_i)$ or $l_1(D) >_1 l_2(D_i \vee A_i)$. We show by induction on β that $D \in \mathcal{T}^D_{M_\alpha} \uparrow (\beta + 1)$ and thus $D \in \mathcal{T}^D_{M_\alpha}$. Then we have $D \in M_{\alpha+1}$ which finishes this case. Let $\beta = 0$. Since there is no ordinal smaller than 0, we know that $l_1(D) > l_1(D_i \vee A_i)$ can only hold if $l_1(D) >_1 l_1(D_i \vee A_i)$. Then all $(A_i \vee D_i) \in M_\alpha$ and all $\neg B_j \in M_\alpha$ and thus $D \in (\mathcal{T}^D_{M_\alpha} \uparrow 1)$, by Definition 4.1. Suppose the property holds for all D with $l(D) = (\alpha, \beta')$, $\beta' < \beta$, and let $l(D) = (\alpha, \beta)$. We know that D satisfies (GDWFi) and, by assumption, for all $(D_i \vee A_i) \in I$ with $l_1(D) > l_1(D_i \vee A_i)$ we have $(D_i \vee A_i) \in (\mathcal{T}^D_{M_\alpha} \uparrow \beta)$ or $(A_i \vee D_i) \in M_\alpha$. Together with $(A_i \vee D_i) \in M_\alpha$ if $l_1(D) >_1 l_2(D_i \vee A_i)$ and $\neg B_j \in M_\alpha$, for all $i = 1, \ldots, n$ and all $j = 1, \ldots, m$, we conclude that $D \in \mathcal{T}^D_{M_\alpha} \uparrow (\beta + 1)$.
(2) If $\neg C \in I$ and $\neg C \in \mathsf{dom}(l_1)$ then C satisfies (GDWFii) and there is an atom $A \in D$ such that for each clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ with $A \in H$ one of the conditions (GDWFiia), (GDWFiia'), or (GDWFiib) holds. Consider such a clause. If (GDWFiib) holds then $(B_1 \vee \ldots \vee B_m) \in I$ and $l_1(\neg C) >_1 l_t(B_1 \vee \ldots \vee B_m)$, $t \in \{1, 2\}$. Then $(B_1 \vee \ldots \vee B_m) \in M_\alpha$. If (GDWFiia') holds then $\neg(A_1 \wedge \ldots \wedge A_n) \in I$ and $l_1(\neg D) >_1 l_2(\neg(A_1 \wedge \ldots \wedge A_n))$. Then also $\neg(A_1 \wedge \ldots \wedge A_n) \in M_\alpha$. If (GDWFiia) holds then there is $\neg(A_1 \wedge \ldots \wedge A_n) \in I$ and $l_1(\neg D) \geq l_1(\neg(A_1 \wedge \ldots \wedge A_n))$. If even $l_1(\neg D) >_1 l_1(\neg(A_1 \wedge \ldots \wedge A_n))$ then $\neg(A_1 \wedge \ldots \wedge A_n) \in M_\alpha$ as well. Otherwise $l_1(\neg(A_1 \wedge \ldots \wedge A_n)) = (\alpha, \beta')$ with $\beta' \leq \beta$. Then $\neg(A_1 \wedge \ldots \wedge A_n)$ satisfies (GDWFii) as well. We can apply the argument again and obtain for each clause that we either have eventually a dependence on an element contained in M_α or an infinite chain of negated conjunctions satisfying (GDWFiia). But then in both cases, by Definition 4.1, $C \in \mathcal{F}^D_{M_\alpha} = \bigcap_{n<\omega} \mathcal{F}^D_{M_\alpha} \downarrow n$ and thus $\neg C \in M_{\alpha+1}$.
(3) If $D \in I$ and $D \in \mathsf{dom}(l_2)$ then D satisfies (GDWFi') and there is a clause $H_1 \vee \cdots \vee H_l \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ such that $\emptyset \neq ((H \cup B) \setminus D') \subseteq D$, $H_k \in D'$ for each $\neg H_k \in I$ with $l_2(D) >_1 l_t(\neg H_k)$, $t \in \{1, 2\}$, $B_j \in D'$

for each $\neg B_j \in I$ with $l_2(D) >_1 l_t(\neg B_j)$, $t \in \{1,2\}$, for all $k = 1,\ldots,l$ and all $j = 1,\ldots,m$, and, for all $i = 1,\ldots,n$, there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$ where $l_2(D) >_2 l_2(D_i \vee A_i)$ or $A_i \in I$ where $l_2(D) >_1 l_s(A_i)$, $s \in \{1,2\}$. Then there is also a clause $H' \leftarrow A_1,\ldots,A_n$ with $H' = H \cup B$ in $DIS(\Pi)$ by definition of $DIS(\Pi)$. We obtain a clause $H'' \leftarrow A_1,\ldots,A_{n'}$ in $\mathsf{ground}(Dis(\Pi, M_\alpha))$ where all $A_i \in M_\alpha$ are removed from the body and we only have $l(D) >_2 l_2(D_{i'} \vee A_{i'}) = (\alpha, \beta')$ with $\beta' < \beta$ for $i' = 1,\ldots,n'$ and non-empty $H'' = H' \setminus D'$. We cannot have removed the clause by definition of $Dis(\Pi, M_\alpha)$ since otherwise D would already occur in M_α in contradiction to our initial assumption. We show by induction on β that $D \in T^D_{Dis(\Pi,M_0)} \uparrow (\beta+1)$, thus $D \in T^{ED}_{M_0}$ and therefore $D \in M_1$. Let $\beta = 0$. Since there is no ordinal smaller than 0 the considered clause is a fact. Then $D \in (T^D_{Dis(\Pi,M_0)} \uparrow 1)$. Suppose the property holds for all D with $l(D) = (\alpha, \beta')$, $\beta' < \beta$, and let $l(D) = (\alpha, \beta)$. We know by assumption for all $(D_i \vee A_i) \in I$ with $l_2(D) >_2 l_2(D_i \vee A_i)$ that $(D_i \vee A_i) \in (T^D_{Dis(\Pi,M_\alpha)} \uparrow \beta)$. Then $D \in T^D_{Dis(\Pi,M_\alpha)} \uparrow (\beta+1)$.
(4) If $\neg C \in I$ and $\neg C \in \mathsf{dom}(l_2)$ then C satisfies (GDWFii') so that $\neg C \in I$ and $C \in \text{EGCWA}(Dis(\Pi, S) \cup S)$, $C \notin S$ and $l_2(\neg C) >_1 l_t(L)$, $t \in \{1,2\}$, if and only if $L \in S$. Thus by definition of $>_1$, $l_t(L) = (\alpha', \beta')$ with $\alpha' < \alpha$ and some β'. If $L \in S$ with $l_t(L) = (\alpha', \beta')$, by induction hypothesis, we know that $L \in M_\alpha$. If $L \in M_\alpha$ then $l_t(L) = (\alpha', \beta')$ with $\alpha' < \alpha$ and $L \in S$. So we have $S = M_\alpha$. Then $C \in \text{EGCWA}(Dis(\Pi, M_\alpha) \cup M_\alpha)$ and $C \notin M_\alpha$. Thus $C \in F^{ED}_{M_\alpha}$ and $\neg C \in M_{\alpha+1}$. ■

Example 4.2. (Example 4.1 continued) In the previous proof we presented how to obtain the levels in general. Thus, we have e.g. $l_1(l \vee r) = l_2(l \vee r) = (0,0)$ by (GDWFi) and (GDWFi'), $l_1(f) = (2,0)$ by (GDWFi), $l_2(\neg p) = (0,0)$ by (GDWFii'), $l_1(\neg e) = (1,0)$ by (GDWFiia') and $l_1(\neg g) = (1,0)$ by (GDWFiia).

The condition (GDWFii') directly refers to EGCWA due to problems with minimal models in the level mapping framework (see Appendix A). We finish this section with a small example dealing with EGCWA.

Example 4.3. Consider the program Π:

$$q \vee r \leftarrow$$
$$q \vee p \leftarrow$$
$$r \vee s \leftarrow$$

We compute M^{ED}_Π. We have $\mathcal{T}^D_{M_0} = \{q \vee r, q \vee p, r \vee s\}$ and $\mathcal{F}^D_{M_0} = \emptyset$. Furthermore, $T^{ED}_{M_0} = \mathcal{T}^D_{M_0}$. The minimal models of $\Pi = DIS(\Pi)$ are $\{q, r\}$, $\{q, s\}$, and $\{r, p\}$. Then $F^{ED}_{M_0} = \{p \wedge q, r \wedge s, p \wedge s\}$. We could try to derive the same directly from the clauses but it is not obvious how this could be done. One attempt could be to take the disjunctive consequences like $p \vee q$ and to conclude that then one should be true and the other one false even though we do not know which one. Then $p \wedge q$ should be false. But this argument does not hold for $q \vee r$ in the example. Additionally, we obtain $p \wedge s$ to be false and there is no evident argument given in the program, at least not in a single clause which allows to draw this conclusion.

5 Disjunctive Well-founded Semantics

The third approach we study is the disjunctive well-founded semantics presented by Brass and Dix in [4]. We use again disjunctive interpretations for representing information even though in [4] a syntactically different method is applied. D-WFS is only defined for (disjunctive) DATALOG programs which are programs whose corresponding language does not have any function symbols apart from (nullary) constants. Thus they correspond to propositional programs and we use the notation Φ from [4] for DATALOG programs.

We recall the operators defining D-WFS. Both map sets of *conditional facts* which are disjunctive clauses without any positive atoms in the body and we start with T_Φ. Given Φ and a set of conditional facts Γ, we have that $T_\Phi(\Gamma) = \{(H \cup \bigcup_i (H_i \setminus \{A_i\})) \leftarrow \neg(B \cup \bigcup_i B_i) \mid \text{there is } H \leftarrow A_1, \ldots, A_n, \neg B \text{ in } \mathsf{ground}(\Phi)$ and conditional facts $H_i \leftarrow \neg B_i \in \Gamma$ with $A_i \in H_i$ for all $i = 1, \ldots, n.\}$ The iteration of T_Φ is given as $T_\Phi \uparrow 0 = \emptyset$, $T_\Phi \uparrow (n+1) = T_\Phi(T_\Phi \uparrow n)$, and $T_\Phi = \bigcup_{n<\omega} T_\Phi \uparrow n$ and yields a fixed point.

The next operator is top-down starting with the previous fixed point also applying the notion of $\mathsf{heads}(S)$ which is the set of all atoms occurring in some head of a clause contained in a given set of ground clauses S: given a set of conditional facts Γ we define $R(\Gamma) = \{H \leftarrow \neg(B \cap \mathsf{heads}(\Gamma)) \mid H \leftarrow \neg B \in \Gamma$, and there is no $H' \leftarrow$ in Γ with $H' \subseteq B$ or there is no $H' \leftarrow \neg B'$ in Γ with $H' \subseteq H$ and $B' \subseteq B$ where at least one $\subseteq$ is proper.$\}$ Note that the second condition forcing one $\subseteq$ to be proper is necessary since otherwise we could remove each conditional fact by means of itself. The iteration of this operator is defined as $R \uparrow 0 = T_\Phi$, $R \uparrow (n+1) = R(R \uparrow n)$ and the fixed point of this operator is called the *residual program* of Φ.

Given the residual program $\mathsf{res}(\Phi)$, the disjunctive well-founded model M_Φ is $M_\Phi = \{D \in EB_\Phi \mid \text{there is } H \leftarrow \text{ in } \mathsf{res}(\Phi) \text{ with } H \subseteq D\} \cup \{\neg D \mid D \in EB_\Phi$ and $\forall D' \in D : D' \notin \mathsf{heads}(\mathsf{res}(\Phi))\}$. Though T_Φ is monotonic, R is not and we cannot generalize the following results to all disjunctive logic programs.

Example 5.1. Recall Π from Example 3.1. It is obvious that Π is also a DATALOG program and we obtain $M_\Pi = \{l \vee r, f, \neg p, \neg c, \neg e, \neg g\}$. Note that M_Π is closed by definition of the model.

In the following, we present the alternative characterization of D-WFS.

Definition 5.1. *Let Φ be a DATALOG program, let I be a model for Φ, and let l be a disjunctive I-partial level mapping for Φ. We say that Φ satisfies (DWF) with respect to I and l if each $D \in \mathsf{dom}(l)$ satisfies one of the following conditions:*

(DWFi) $D \in I$ and there is a clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Phi)$ with $H \subseteq D$ such that there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$,

$l(D) > l(D_i \vee A_i)$, and $l(D) \succ l(D_i \vee A_i)$ if $l(D) >_1 l(D_i \vee A_i)$, for all $i = 1, \dots, n$, and $\neg B_j \in I$ and $l(D) >_1 l(B_j)$ for all $j = 1, \dots, m$.
(DWFii) $\neg D \in I$ and for each clause $H \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m$ in $\mathsf{ground}(\Phi)$ *with $A \in H$ and $A \in D$ (at least) one of the following conditions holds:*
(DWFiia) $\neg A_i \in I$ and $l(D) \geq l(A_i)$.
(DWFiib) $D' \in I$ with $D' \subseteq B$ and $l(D) >_1 l(D')$.
(DWFii') $\neg D \in I$ and for each conditional fact $H \leftarrow \neg B$ in T_Φ with $A \in H$ and $A \in D$ (at least) one of the following conditions holds:
(DWFiia') there is $H' \leftarrow \neg B'$ in $R \uparrow \alpha$ with $H' \subset H$ and $B' \subseteq (B \setminus D')$ where $A \notin H'$, $B_j \in B$, $\neg B_j \in I$, and $l(D) >_1 (l(B_j) + 1)$ for all $B_j \in D'$, and $l(D) >_1 (\alpha, \beta)$ for some β.
(DWFiib') $D' \in I$ with $D' \subseteq B$ and $l(D) >_1 l(D')$.

Theorem 5.1. *Let Φ be a (disjunctive) DATALOG program with disjunctive well-founded model M. Then, in the disjunctive knowledge ordering, M is the greatest model amongst all models I for which there exists a disjunctive I-partial level mapping l for Φ such that Φ satisfies (DWF) with respect to I and l.*

Proof. Let M be the disjunctive well-founded model of Φ. We define the disjunctive M-partial level mapping l in the following way: If $D \in M$ then $l(D) = (\alpha, \beta)$ where α is the least ordinal such that $H \leftarrow$ in $R \uparrow \alpha$ with $H \subseteq D$ and β is the least ordinal such that the corresponding conditional fact $H \leftarrow \neg B$ in $T_\Phi \uparrow (\beta + 1)$. If $\neg D \in M$ then $l(D) = (\alpha, 0)$ where α is the least ordinal such that for each $A \in D$ there is no conditional fact $H \leftarrow \neg B$ in $R \uparrow \alpha$ with $A \in H$. All other values remain undefined. Note that we do not assign any limit ordinals apart from 0 by means of the definitions of the operators and the restriction to DATALOG programs. By definition of T_Φ and R, we conclude that l is well-defined, i.e. if $D \in M$ or $\neg D \in M$ then $D \in \mathsf{dom}(l)$. We show that Φ satisfies (DWF) with respect to M and l and consider two cases:
(1) Let D be in M and $l(D) = (\alpha, \beta)$. We have $H \leftarrow$ in $R \uparrow \alpha$ and $H \leftarrow \neg B$ in $T_\Phi \uparrow (\beta + 1)$ with $H \subseteq D$ by definition of l. By definition of R, for all $B_j \in B$, $l(B_j) = (\alpha', 0)$ with $\alpha' < \alpha$ and $\neg B_j \in M$. By definition of T_Φ, we can unfold the derivation of the conditional fact, so there are $H' \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m$ in $\mathsf{ground}(\Phi)$ and conditional facts $H_i \leftarrow \neg B'_i$ in $(T_\Phi \uparrow \beta)$ with $A_i \in H_i$, for all $i = 1, \dots, n$, and $B = (\bigcup_i B'_i \cup \{B_1, \dots, B_m\})$. We know that $l(D) >_1 l(B_j)$ for all $j = 1, \dots, m$ since all $B_j \in B$. Furthermore, $H_i = D_i \vee A_i$ for some D_i and since $H_i \leftarrow \neg B'_i$ in $(T_\Phi \uparrow \beta)$, we have $H_i \in M$ and thus $(D_i \vee A_i) \in M$ and $l(D) \succ l(D_i \vee A_i)$. Since $l(D) >_1 l(B')$ for all $B' \in B'_i$, $i = 1, \dots, n$, we conclude that also $l(D) > l(D_i \vee A_i)$ which shows that (DWFi) is satisfied.
(2) Alternatively, $\neg D \in M$ and $l(D) = (\alpha, 0)$. By definition of l, α is the least ordinal such that for each $A \in D$ there is no conditional fact $H \leftarrow \neg B$ in $R \uparrow \alpha$ with $A \in H$. Thus consider any conditional fact $H \leftarrow \neg B$ in T_Φ with $A \in H$ and $A \in D$. Then this conditional fact also occurs in $R \uparrow 0$ by definition of R and is thus removed by an iteration step $R \uparrow \alpha'$, $\alpha' < \alpha$. By definition of R, a conditional fact $H \leftarrow \neg B$ is removed from a set Γ if there is $H' \leftarrow$ in Γ with

$H \subseteq B$, or there is $H' \leftarrow \neg B'$ in Γ with $H' \subseteq H$ and $B' \subseteq B$ where at least one $\subseteq$ is proper.

(a) Suppose that $H \leftarrow \neg B$ is removed in the iteration step $R \uparrow \alpha'$ because there is $H' \leftarrow$ in $R \uparrow (\alpha' - 1)$ with $H' \subseteq B$. Then $H' \in M$. By definition of l, we have $l(H') = (\alpha'', \beta')$ for some β' and $\alpha'' < \alpha'$ and thus $l(D) >_1 l(H')$. Hence (DWFiib') is satisfied.

(b) Otherwise, if $H \leftarrow \neg B$ is removed in the iteration step $R \uparrow \alpha'$, because there is $H' \leftarrow \neg B'$ in $R \uparrow (\alpha' - 1)$ with $H' \subseteq H$ and $B' \subseteq B$ where at least one $\subseteq$ is proper, then we also have a conditional fact $H \leftarrow \neg(B \cup D')$ in $R \uparrow 0$ with $\neg B'_j \in I$ and $l(B'_j) = (\alpha'', \beta'')$, $\alpha'' < \alpha'$, for all $B'_j \in D'$. Then $l(D) >_1 (l(B'_j)+1)$ and $l(D) >_1 (\alpha', \beta')$ for some β'. Without loss of generality we assume that $A \notin H'$, otherwise there still would be a conditional fact contained in $R \uparrow \alpha'$ with A in the head which is also eliminated by one of the conditions in the definition of R, and in both cases this implies that either (DWFiia') or (DWFiib') is satisfied. We conclude that (DWFiia') is satisfied.

Conversely, we show that if I is a model of Φ and l a disjunctive I-partial level mapping such that Φ satisfies (DWF) with respect to I and l then $I \subseteq M_\Phi$. We show via induction on α for $l(D) = (\alpha, \beta)$ that whenever $D \in I$ then $H \leftarrow$ in $R \uparrow \alpha$ with $H \subseteq D$ and whenever $\neg D \in I$ then for all $A \in D$ there is no conditional fact $H \leftarrow \neg B$ in $R \uparrow \alpha$ with $A \in H$. This suffices to show that $D \in M_\Phi$, $\neg D \in M_\Phi$ respectively.

Let $\alpha = 0$. We have to consider three cases.

(1) Let $D \in I$. By (DWFi) there is a clause $H \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m$ in $\mathsf{ground}(\Phi)$ with $H \subseteq D$ such that there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$, $l(D) > l(D_i \vee A_i)$, and $l(D) \succ l(D_i \vee A_i)$ if $l(D) >_1 l(D_i \vee A_i)$, for all $i = 1, \dots, n$, and $\neg B_j \in I$ and $l(D) >_1 l(B_j)$ for all $j = 1, \dots, m$. Since there is no ordinal smaller than 0, we know that $>_1$ cannot hold and (DWFi) simplifies to that there is a clause $H \leftarrow A_1, \dots, A_n$ in $\mathsf{ground}(\Phi)$ with $H \subseteq D$ such that there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$ and $l(D) >_2 l(D_i \vee A_i)$ for all $i = 1, \dots, n$. We show by induction on β that $(H \cup \bigcup_i D_i) \leftarrow$ in $T_\Phi \uparrow (\beta + 1)$ and thus $(H \cup \bigcup_i D_i) \leftarrow$ in $R \uparrow 0$. This suffices since $(H \cup \bigcup_i D_i) \subseteq D$.

Let β be 0. Since there is no ordinal smaller than 0, the considered clause is a fact, $H \leftarrow$ in $T_\Phi \uparrow 1$ and all D_i are empty.

Suppose that the claim holds for all $D \in I$ with $l(D) = (0, \beta')$, $\beta' \leq \beta$ and let $D \in I$ with $l(D) = (0, \beta + 1)$. We know that D satisfies the simplified (DWFi). By assumption, for all $A_i \vee D_i$ we have a conditional fact $H'_i \leftarrow$ in $T_\Phi \uparrow (\beta + 1)$ with $H'_i \subseteq (D_i \cup \{A_i\})$. If $A_i \notin H'_i$ for one i then we have $H'_i \subseteq D_i$ and since $D_i \subseteq D$ we obtain that $H'_i \leftarrow$ in $T_\Phi \uparrow (\beta + 1)$ and thus also in $T_\Phi \uparrow (\beta + 2)$. Otherwise $A_i \in H_i$ for all $i = 1, \dots, n$ and $H'_i = (D'_i \cup \{A_i\})$ with $D'_i \subseteq D_i$. By definition of T_Φ we then have a fact $(H \cup \bigcup_i D_i) \leftarrow$ in $T_\Phi \uparrow (\beta + 2)$ with $(H \cup \bigcup_i D_i) \subseteq D$ which finishes this case.

(2) Let $\neg D \in I$ and D satisfies (DWFii'). Then for each conditional fact $H \leftarrow \neg B$ in T_Φ with $A \in H$ and $A \in D$ we have that (DWFiia') or (DWFiib') holds. Since there is no ordinal smaller than 0, neither (DWFiia') nor (DWFiib') can hold and there cannot be any such conditional fact in T_Φ which finishes this case.

(3) Let $\neg D \in I$ and D satisfies (DWFii). Since we know that there is no ordinal smaller than 0, (DWFiib) cannot hold and by (DWFiia) for each clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Phi)$ with $A \in H$ and $A \in D$ we have $\neg A_i \in I$ and $l(D) >_2 l(A_i)$. Consider such a clause. By definition of l, we know that A_i also satisfies either (DWFii) or (DWFii') but we already know that (DWFii') can only be satisfied if there is no particular conditional fact and the same holds for any clause and (DWFiib). Thus, A_i satisfies (DWFii) and (DWFiia) holds for each occurring particular clause. If there is no such clause (conditional fact) with A_i in the head at all then the initially considered clause could not have been applied for calculating a conditional fact wrt. D. So there is at least one clause with satisfies (DWFiia) and we can re-apply the argument. Thus we derive an infinite chain of atoms which are false in I and satisfy (DWFiia) which is not possible since we are dealing with DATALOG programs. Hence, there cannot be any clause in $\mathsf{ground}(\Phi)$ with $A \in H$ and $A \in D$ and thus no such conditional fact either.

Assume for all D with $l(D) = (\alpha', \beta')$, for arbitrary β' and $\alpha' \leq \alpha$, that whenever $D \in I$ then $H \leftarrow$ in $R \uparrow \alpha'$ with $H \subseteq D$ and whenever $\neg D \in I$ then for all $A \in D$ there is no conditional fact $H \leftarrow \neg B$ in $R \uparrow \alpha'$. We are going to show this property for D with $l(D) = (\alpha + 1, \beta)$ for some β, and we consider again three cases.

(1) Let $D \in I$. By (DWFi) there is a clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Phi)$ with $H \subseteq D$ such that there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$, $l(D) > l(D_i \vee A_i)$, and $l(D) \succ l(D_i \vee A_i)$ if $l(D) >_1 l(D_i \vee A_i)$, for all $i = 1, \ldots, n$, and $\neg B_j \in I$ and $l(D) >_1 l(B_j)$ for all $j = 1, \ldots, m$. We show by induction on β that $(H \cup \bigcup_i D_i) \leftarrow \neg B_1, \ldots, \neg B_m, \neg B_{m+1}, \ldots, \neg B_r$ in $T_\Phi \uparrow (\beta + 1)$ with $(H \cup \bigcup_i D_i) \subseteq D$ and $\neg B_j \in M_\Phi$ and $l(D) >_1 l(\neg B_j)$ for all $j = 1, \ldots, r$. Then $(H \cup \bigcup_i D_i) \leftarrow \neg B_1, \ldots, \neg B_m, \neg B_{m+1}, \ldots, \neg B_r$ in T_Φ. Since $l(D) >_1 l(\neg B_j)$, for all $j = 1, \ldots, r$, we know by induction hypothesis that all $B_j \notin \mathsf{heads}(R \uparrow \alpha)$ and thus none of these appear in the body of the conditional fact occurring in $R \uparrow (\alpha + 1)$. Then $(H \cup \bigcup_i D_i) \leftarrow$ in $R \uparrow (\alpha + 1)$ which finishes this case because $(H \cup \bigcup_i D_i) \subseteq D$.

Let β be 0. Consider there are positive atoms in the body of the clause. Since there is no ordinal smaller than 0, we know that $l(D) > l(D_i \vee A_i)$ for all $i = 1, \ldots, n$, can only be satisfied by $l(D) >_1 l(D_i \vee A_i)$. Thus for all these atoms the additional condition $l(D) \succ l(D_i \vee A_i)$ from (DWFi) has to hold which is not possible and there cannot be any positive atoms in the body. Then $H \leftarrow \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Phi)$ and also in $T_\Phi \uparrow 1$.

Suppose that the property holds for all D with $l(D) = (\alpha, \beta')$, $\beta' \leq \beta$. We show that it holds for D with $l(D) = (\alpha, \beta + 1)$. For all $D_i \vee A_i$ we know $l(D) > l(D_i \vee A_i)$. If $l(D) >_1 l(D_i \vee A_i)$ then $l(D) \succ l(D_i \vee A_i)$. Otherwise $l(D) >_2 l(D_i \vee A_i)$ and, by assumption, respectively by induction hypothesis in the former case, there is a conditional fact $H'_i \leftarrow \neg B'_i$ in $T_\Phi \uparrow (\beta + 1)$ with $H'_i \subseteq (D_i \cup \{A_i\})$. If $A_i \notin H'_i$ for one i then we have $H'_i \subseteq D_i$. Since $D_i \subseteq D$ we obtain that $H'_i \leftarrow \neg B'_i$ in $T_\Phi \uparrow (\beta + 1)$ and thus also in $T_\Phi \uparrow (\beta + 2)$. Otherwise $A_i \in H_i$ for all $i = 1, \ldots, n$ and $H'_i = (D'_i \cup \{A_i\})$ with $D'_i \subseteq D_i$. By definition of T_Φ, we then have a conditional fact $(H \cup \bigcup_i D_i) \leftarrow \neg(\bigcup_i B'_i \cup \{B_1,\ldots,B_m\})$ in $T_\Phi \uparrow (\beta+2)$ with $(H \cup \bigcup_i D_i) \subseteq D$.

(2) Let $\neg D \in I$ and D satisfies (DWFii'). Then for each conditional fact $H \leftarrow \neg B$ in T_Φ with $A \in H$ and $A \in D$ we know that (DWFiia') or (DWFiib') holds. Consider such a conditional fact. If (DWFiia') holds then there is $H' \leftarrow \neg B'$ in $R \uparrow \alpha'$ with $H' \subset H$ and $B' \subseteq (B \setminus D')$ where $A \notin H'$, $B_j \in B$, $\neg B_j \in I$, and $l(D) >_1 (l(B_j)+1)$ for all $B_j \in D'$, and $l(D) >_1 (\alpha',\beta')$ for some β'. Then for all $B_j \in D'$, $B_j \notin \mathsf{heads}(R \uparrow (\alpha-1))$. Thus $H \leftarrow \neg(B \setminus D')$ in $R \uparrow \alpha$. Then by definition of R, $H \leftarrow \neg(B \setminus D')$ is not contained in $R \uparrow (\alpha+1)$ since $A_i \notin H'$. If (DWFiib') holds then $D' \in I$ with $D' \subseteq B$ and $l(D) >_1 l(D')$. By induction hypothesis, we know that $H \leftarrow$ in $R \uparrow \alpha'$ with $H \subseteq D'$ and $\alpha' \leq \alpha$ and therefore also in $R \uparrow \alpha$. Then by definition of R, $H \leftarrow \neg B$ is not contained in $R \uparrow (\alpha+1)$ as well. Thus there is no conditional fact $H \leftarrow \neg B$ with $A \in H$ and $A \in D$ in $R \uparrow (\alpha+1)$ which finishes this case.

(3) Let $\neg D \in I$ and D satisfies (DWFii). Then for each clause $H \leftarrow A_1,\ldots,A_n$, $\neg B_1,\ldots,\neg B_m$ in $\mathsf{ground}(\Phi)$ with $A \in H$ and $A \in D$ we have that (DWFiia) or (DWFiib) holds. Consider such a clause:

(a) If (DWFiib) holds then $D' \in I$ with $D' \subseteq B$ and $l(D) >_1 l(D')$. If this clause is used to add a conditional fact to T_Φ then this conditional fact contains still all $\neg B_j$, $j=1,\ldots,m$. By induction hypothesis, we know that there is $H \leftarrow$ in $R \uparrow \alpha'$, $\alpha' \leq \alpha$, with $H \subseteq D'$ and thus also in $R \uparrow \alpha$. Then this conditional fact with A in the head does not occur in $R \uparrow (\alpha+1)$ by definition of R. Otherwise this clause is not used to add a conditional fact to T_Φ and there is no resulting conditional fact in $R \uparrow (\alpha+1)$ either.

(b) If (DWFiia) holds then $\neg A_i \in I$ and $l(D) \geq l(A_i)$. We consider two cases. A_i satisfies (DWFii'). Then for each conditional fact $H \leftarrow \neg B$ in T_Φ with $A_i \in H$ we know that (DWFiia') or (DWFiib') holds. Consider such a conditional fact. If (DWFiia') holds then there is $H' \leftarrow \neg B'$ in $R \uparrow \alpha'$ with $H' \subset H$ and $B' \subseteq (B \setminus D')$ where $A \notin H'$, $B_j \in B$, $\neg B_j \in I$, and $l(D) >_1 (l(B_j)+1)$ for all $B_j \in D'$, and $l(D) >_1 (\alpha',\beta')$ for some β'. If we apply this conditional fact in one step of T_Φ to the clause mentioned above then the resulting conditional fact in T_Φ has to contain $H \setminus \{A_i\}$ as a subset of the head and $\neg B$ as a subset of the negative literals in the resulting body. But then this resulting conditional fact is removed by $H' \leftarrow \neg B'$ in $R \uparrow \alpha$ and thus does not occur in $R \uparrow (\alpha+1)$. We derive that in this case (DWFiia') is also satisfied for A. If (DWFiib') holds for A_i then $D' \in I$ with $D' \subseteq B$ and $l(A_i) >_1 l(D')$. If we use this conditional fact $H \leftarrow \neg B$ in T_Φ with $A_i \in H$ to derive a conditional fact then, by means of T_Φ, the clause mentioned above, and by substituting A_i, $\neg B$ is a subset of the body of the resulting conditional fact. But since $D' \in I$ with $D' \subseteq B$ and $l(A_i) >_1 l(D')$ we know by induction hypothesis that there is $H' \leftarrow$ in $R \uparrow \alpha$ with $H' \subseteq D'$ and thus the resulting conditional fact does not occur $R \uparrow (\alpha+1)$ and therefore A satisfies also (DWFiib'). Alternatively, A_i satisfies (DWFii). Then for each clause $H \leftarrow A'_1,\ldots,A'_n,\neg B'_1,\ldots,\neg B'_m$ in $\mathsf{ground}(\Phi)$ with $A_i \in H$ we have that (DWFiia) or (DWFiib) holds. Consider such a clause. If (DWFiib) holds then $D' \in I$ with $D' \subseteq B'$ and $l(A_i) >_1 l(D')$. If this clause is used to derive a conditional fact in T_Φ then we can apply the very same argument as in case of (DWFiib') and obtain that A also satisfies (DWFiib'). Alternatively, (DWFiia) holds and $\neg A'_{i'} \in I$ and $l(A_i) \geq l(A'_{i'})$. Then we can re-apply the argument and for each clause we derive either eventually a dependency on (DWFii') or we have an infinite chain of atoms which are false in I and satisfy (DWFiia). But an infinite chain is impossible by means of our restriction to DATALOG programs. Thus, also in this case we do not have a conditional fact in $R \uparrow (\alpha + 1)$ with A in the head which finishes the case and the induction step. ■

Example 5.2. (Example 5.1 continued) In one direction of the previous proof we have seen how to calculate the level mapping. We obtain e.g. $l(f) = (2, 0)$ by (DWFi), $l(p) = (1, 0)$ by (DWFiia'), $l(c) = (1, 0)$ by (DWFiib) and $l(e) = (1, 0)$ by (DWFiia).

We demonstrate the reasons for some of the conditions with the examples below.

Example 5.3. We start with the following program Φ.

$$\begin{aligned} p &\leftarrow r, \neg q \\ r &\leftarrow \neg s \\ q \vee s &\leftarrow \end{aligned}$$

We obtain $T_\Phi = \{p \vee q \leftarrow \neg s, \neg q, r \leftarrow \neg s, q \vee s \leftarrow\}$. Then $R \uparrow 1 = \{r \leftarrow \neg s, q \vee s \leftarrow\}$ which is also the fixed point and we have $M_\Phi = \{q \vee s, \neg p\}$. The situation is similar to that of the strong well-founded semantics. We derive p to be false but both, q and r, remain undefined and (DWFii) alone is not sufficient for the characterization. Instead, we need a more general case which is (DWFiib') covering the combination of the negative literals from several clauses.

Finally, we will present arguments for the introduction of $\succ$ as additional relation.

Example 5.4. Let Φ be the following.

$$\begin{aligned} a &\leftarrow \\ b &\leftarrow a \\ c &\leftarrow b, \neg d \end{aligned}$$

We have $T_\Phi = \{a \leftarrow, b \leftarrow, c \leftarrow \neg d\}$ and $R \uparrow 1 = \{a \leftarrow, b \leftarrow, c \leftarrow\}$. Then $M_\Phi = \{a, b, c, \neg d\}$ and we obtain $l(a) = (0, 0)$, $l(b) = (0, 1)$, $l(c) = (1, 2)$, and $l(d) = (0, 0)$. Obviously, c is true and satisfies (DWFi). However, we could set $l(c) = (1, 0)$ and remove the part of the condition referring to $\succ$ and this would hold as well. But then the mapping does no longer correspond to the construction used to define it in the first part of the proof of Theorem 5.1 and it would not be possible to show the other direction of the equivalence which is based on this construction.

6 Discussions

It was already shown in [4] that D-WFS and GDWFS satisfy five program transformation rules while SWFS does not, and that GDWFS always derives more or equal knowledge than D-WFS [5]. However, there is no similar result for D-WFS and SWFS since they are incomparable with respect to the derived knowledge (cf. our main example: SWFS derives $\neg b$ while D-WFS concludes $\neg p$).

We will now further compare the semantics on the basis of our characterizations. We will in particular attempt to obtain some insights into good general criteria for a well-founded semantics for disjunctive programs.

A main advantage of level mapping characterizations is the separation of positive and negative information. One key insight which can be drawn from our investigations is that any characterization basically states that a true disjunction D satisfies the following scheme with respect to the model I and the program Π.

> $D \in I$ and there is a clause $H \leftarrow A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ in $\mathsf{ground}(\Pi)$ with $H \subseteq D$ such that there is $D_i \subseteq D$ with $(D_i \vee A_i) \in I$, $l(D) > l(D_i \vee A_i)$, for all $i = 1, \ldots, n$, and $\neg B_j \in I$ and $l(D) > l(B_j)$ for all $j = 1, \ldots, m$.

We can see that this corresponds in general to (SWFia) from Definition 3.4, to (GDWFi) from Definition 4.2, and to (DWFi) from Definition 5.1. We only have to consider that the relation $>$ is technically not sufficient and that we sometimes apply a more precise order. Nevertheless, in all cases we obtain levels such that $l(D)$ is greater with respect to the specific ordering. There are further differing details. For (SWF), we have to abstract additionally from the notion of derivation sequences and their children, and there is also (SWFib) which arises from proof-theoretical treatments. In case of (GDWF) we have additionally a condition (GDWFi') but that is the part (corresponding to T_{Π}^{D}) which derives more knowledge than the well-founded semantics and should thus not be an intended result for a well-founded semantics for disjunctive programs. We claim that the condition given above is the 'disjunctive' version of (WFi) from Definition 2.1 and we propose it to be a condition for any semantics aiming to extend the well-founded semantics to disjunctive programs.

If we look for adequate extensions of (WFii) to disjunctive programs then we see that the conditions for negative information differ more. A straightforward extension to disjunctive programs is given in Definition 5.1 with (DWFii) containing appropriate cases (DWFiia) and (DWFiib). In case of (DWFiia), for SWFS we have (SWFiia') which somehow approximates it, ignoring derivation sequences, but we also have (SWFiia) which refers to $A_i \vee D$ being false instead of just A_i. Moreover, we have (GDWFiia) and (GDWFiia') corresponding to (DWFiia) but possibly being based on indefinite information. We only have two cases because of the construction applied for GDWFS, and more generally, one statement suffices to guarantee that $\geq$ holds wrt. some order. For (DWFiib) the situation appears to be simpler, because we have (SWFiib') and (GDWFiib) as corresponding statements, abstracting from minor technical details. Unfortunately, since positive information may be indefinite, it is also possible to obtain a correspondence to (WFiib) which results from several clauses (consider the program $\Pi = \{p \vee q \leftarrow;\ r \leftarrow s, \neg p;\ s \leftarrow \neg q\}$ where $\neg r$ is derivable). This is covered by (SWFiic), (GDWFii'), and (DWFiib'). Still, this does not cover the whole characterization for any of the semantics. (SWFiib") extends (SWFiib') to include particular atoms from the head. (GDWFii') is in fact much more powerful by means of the EGCWA and allows for deriving more knowledge difficult to characterize in a clause-based approach. In case of D-WFS we also have (DWFiia') which resolves the elimination of non-minimal clauses, a feature not contained in SWFS and also covered by (GDWFii') for GDWFS.

Summarising, it is obvious (and certainly expected) that it is in the derivation of negative information where the semantics differ wildly. All characterizations contain extensions of (WFii), but contain also additional non-trivial conditions some of which are difficult to capture within level mapping characterizations. The obtained uniform characterizations thus display in a very explicit manner the very different natures of the different well-founded semantics – there is simply not enough resemblance between the approaches to obtain a coherent picture. We can thus, basically, only confirm in a more formal way what has been known beforehand, namely that the issue of a good definition of well-founded semantics for disjunctive logic programs remains widely open. We still believe, though, that our structured approach delivers structural insights which can help to guide the quest.

7 Conclusions and Further Work

We have characterized three of the extensions of the well-founded semantics to disjunctive logic programs. It has been revealed that these characterizations are non-trivial and we have seen that they share a common derivability for true disjunctions. The conditions for deriving negative information however vary a lot. Some parts of the characterizations are common extensions of conditions used for the well-founded semantics while others cover specific deduction mechanisms occurring only in one semantics. We have obtained some structural insights into the differences and similarities of proposals for disjunctive well-founded semantics, but the main conclusion we can draw is a negative one: Even under our formal approach which provides uniform characterizations of different semantics, the different proposals turn out to be too diverse for a meaningful comparison. The quest for disjunctive well-founded semantics thus remains widely open. Our uniform characterizations may provide arguments for approaching the quest in a more systematic way.

In this paper, we covered only those of the well-founded semantics which a priory appeared to be the most important and promising ones. Obviously, further insights could be obtained from considering also the well-founded disjunctive semantics WFDS [20] and the well-founded semantics with disjunction WFS_d [1]. Such a treatment would also include the stationary semantics [14] and the static semantics [16] where the latter is closely related to D-WFS, and also WF 3 [3], an extension to GDWFS. The semantics we considered interpret disjunctions exclusively whenever possible, but there are also some semantics employing inclusive disjunctions (see [5]) including variants of the semantics presented here. These could also be taken into consideration.

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A Minimal Model Semantics

In this appendix we discuss some difficulties we encountered with the level mapping characterization of the minimal model semantics [12], which is in fact needed implicitly for the EGCWA in case of the GDWFS. Indeed, Definition 4.2 contains a rather unspecific condition referring directly to EGCWA and not to clauses. We recall in the following the characterization of minimal models from [10].

Theorem A.1. *([10]) Let Π be a definite disjunctive program. Then a model M of Π is a minimal model of Π if and only if there exists a total level mapping $l : B_\Pi \to \alpha$ such that for each $A \in M$ exists a clause $A \vee H_1 \vee \cdots \vee H_l \leftarrow A_1, \ldots, A_n$ in $\mathsf{ground}(\Pi)$ with $A_i \in M$, $H_k \notin M$, and $l(A) > l(A_i)$ for all $i = 1, \ldots, n$ and all $k = 1, \ldots, l$.*

Unfortunately, as it turned out, this characterization is too strong, meaning that there are programs with a minimal model that does not satisfy this condition.

Example A.1.

$$\begin{aligned} a \vee b &\leftarrow \\ a &\leftarrow b \\ b &\leftarrow a \end{aligned}$$

This program has only one minimal model $\{a, b\}$, so according to the condition above, the first clause cannot be used since both atoms in the head are true. With the remaining two clauses we cannot have a level mapping satisfying the given condition since we must have $l(a) > l(b)$ and $l(b) > l(a)$ which is not possible.

A first idea to fix the problem was to modify the condition to let all atoms in the head of the specific clause for A either to be false or of level strictly greater than A. But this condition is too weak.

Example A.2.

$$\begin{aligned} a \vee b &\leftarrow \\ a \vee c &\leftarrow \end{aligned}$$

This program has two minimal models: $\{a\}$ and $\{b, c\}$. But $\{a, b\}$ is also a model and with $l(a) = 0$ and $l(b) = 1$ we have a mapping satisfying the modified condition, even though $\{a, b\}$ is not minimal.

The next attempt contains additionally a further clause for each $A \in M$ with A being the only true atom in the head but without any restrictions regarding the level.

Example A.3. We show with the following program Π that this does not work either.

$$\begin{aligned} p(0) &\leftarrow \\ p(s(X)) \vee p(X) &\leftarrow \\ p(X) &\leftarrow p(s(s(X))) \end{aligned}$$

A minimal model is e.g. $\{p(s^{2n}(0)) \mid n \geq 0\}$ with $l(p(s^n(0))) = 0$. However, B_Π is also a model and, by $l(p(s^n(0))) = n$, satisfies this further modified condition which is therefore also too weak.

Since the levels for the minimal model are smaller than for the Herbrand basis we finally try to allow instead only minimal mappings which is natural in so far that the assignments given in the proofs of all characterizations presented up to now are minimal. Then the Herbrand basis cannot be a model satisfying the condition in the example above. But in the following we also present a counterexample to this idea.

Example A.4.

$$\begin{aligned} b \vee c &\leftarrow \\ c &\leftarrow b \\ d &\leftarrow c \\ e &\leftarrow \\ f &\leftarrow e \\ d &\leftarrow f \end{aligned}$$

Given that program Π, we have a model $\{b, c, d, e, f\}$ with $l(b) = 0$, $l(c) = 1$, $l(d) = 1$, $l(e) = 0$, and $l(f) = 1$ and there is no model with a mapping such that all values are smaller or equal with one value being strictly smaller. However, we have a minimal model $\{c, d, e, f\}$ with $l_1(b) = 0$, $l_1(c) = 0$, $l_1(d) = 2$, $l_1(e) = 0$, and $l_1(f) = 1$ even though $l(c) > l_1(c)$ and $l(d) < l_1(d)$ and the mapping is not smaller.

Thus, the best possible result we may get is the following.

Corollary A.1. *Let Π be a definite disjunctive program. If there exists a total level mapping $l : B_\Pi \to \alpha$ such that for each $A \in M$ exists a clause $A \vee H_1 \vee \cdots \vee H_l \leftarrow A_1, \ldots, A_n$ in $\mathsf{ground}(\Pi)$ with $A_i \in M$, $H_k \notin M$ or $l(H_k) > l(A)$, and $l(A) > l(A_i)$, for all $i = 1, \ldots, n$ and all $k = 1, \ldots, l$, then M is a minimal model of Π.*

This is of course no equivalent characterization. It just states that minimal models induce a level mapping which does not help us to simplify a characterization involving minimal models. The same problem occurs for disjunctive stable models ([15]), the extension of minimal model semantics to non-definite programs. Summing up, it seems that level mapping characterizations are not easy to extend to disjunctive programs in general.